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Square-Root Operator Quantization and Nonlocality: A Review

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A square-root-operator formalism is developed for quantum systems described with nonrelativistic and relativistic equations of motion. Spectral representation for Green's functions are designed for particles with spin 0, with the implication of its generalization to other spin values. Nonlocal operators suggest that a duality exists between physical particles and dual partners, which are tachyonic mathematical particles. It is shown that nonlocal operators result naturally from square-root operators, with the implication that microcausality holds only asymptotically. Applications help enlighten the formalism in order to envisage realistic situations with Schrödinger equations, Higgs fields, vacuum fluctuations, extra-dimensional methods in the potential theory, and electromagnetic interactions of extended charges and their consequences. It turns out that the innermost structure of these extended charges is associated with nonlocal photon propagators. It is shown that the propagator arisen from the charged torus potential consists of two different parts: a nonlocal photon propagator and a propagator of neutrino-like particles, which is described by square-root-operator equation. We examine the potential of the torus and its propagator as the appearance of superfields in terms of the photon and the massless fermion (photino).

The concept of extended and nonlocal objects as well as their interactions receives rising interest in physics and mathematics. String theory is an example for which far-reaching, detailed studies exist (Green *et al.*, 1986; Polchinski, 1998). Nonlocality is a feature of quantum field theory (QFT), which was already realized in its infant stage when the ultraviolet divergence terms arose. If we characterize the electron size with a parameter ℓ, its classical field energy is $\sim e^2/\ell$, whereas in QFT it is $\sim e^2 m_e \log \ell m_e$. For a pointlike electron with $\ell \to 0$, both values diverge and thus lose their meaning. One possible way out of this divergence problem is to take refuge in nonlinearity and, in particular, nonlocality. The

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consequence is that differential equations become integro-differential equations. In case of the Schrödinger equation, the potential operator becomes an integral operator.

If we use a relativistic relation between momentum and energy, classically

$$E = \sqrt{p^2 + m^2}, \tag{1}$$

as operator[5]

$$i\frac{\partial}{\partial t} = \sqrt{m^2 - \nabla^2}. \tag{2}$$

The square-root operator is not defined in r space, and a Fourier transformation (FT) into a k-space representation is used in common practice.

The assumption of pointlike masses and charges also cause essential difficulties, since pointlike is effectively represented only by a distribution, in particular, by a Dirac δ function,

$$\rho_m(x) = m\delta(x), \qquad \rho_e(x) = e\delta(x). \tag{3}$$

This representation already yields the key paths for its handling in practice. In Section 10 we summarize the salient features of distributions with important results and relations, which shall be used throughout this text.

At this stage, the mathematical developments are rather complete, but their practical application and the resulting physical interpretation are less well established. In case of integral equations, proofs of convergence and efficient representations in terms of Fourier series are still popular topics in mathematical physics. Among choices of relativistic form factors (Efimov, 1977, and Namsrai, 1986), replacements like

$$\frac{1}{m^2 - p^2 - i\varepsilon} \to \frac{K_\ell^2(-p^2\ell^2)}{m^2 - p^2 - i\varepsilon} \tag{4}$$

are often used. In these notes we shall outline and discuss several such schemes for problems associated with nonlocality in particle physics.

Already in the early developments of quantum mechanics occur square-root operators. In particular, it was the relativistic relation between energy and momentum in a coordinate space representation that hindered its use (Weyl, 1927). A review of the early and later works are contained in Smith (1993). Today, in bound-state problems of two- and three-quark systems the Salpeter equation is often used (Castorina *et al.*, 1984; Friar and Tomusiak, 1984; Nickisch *et al.*, 1984). Furthermore, problems associated with binding in very strong fields (Hardekopf and Sucher, 1985; Papp, 1985), string theory (Fiziev, 1985; Kaku, 1988), and

[5] Units with $c = 1$ and $\hbar = 1$ are used throughout.

astrophysical black holes (Berezin, 1997a,b; Berezin *et al.*, 1998) are application areas. Green's functions for differential equations of infinite order like

$$\sqrt{m^2 - \Box}D(x) = -\delta^{(4)}(x) \tag{5}$$

are treated in Namsrai (1998). There exist essential differences to Green's functions of the second-order d'Alembert operator. This differences is the nonlocality that describes self-interactions of particles. As a typical example consider the free cases

$$i\frac{\partial\psi(x)}{\partial t} = \sqrt{m^2 - \nabla^2}\psi(x) \tag{6}$$

and

$$\sqrt{m^2 - \Box}\varphi(x) = 0. \tag{7}$$

A binomial expansion of the square root yields formally an integral equation

$$\int d^4y\, K_{\ell_m}(x-y)\varphi(y) = 0$$

with a kernel containing the Dirac δ function

$$K_{\ell_m}(x) = \left[m + \frac{m}{2}\sum_{n=0}^{\infty} C_n\left(-\ell_m^2\right)^{n+1}\Box^{n+1}\right]\delta^{(4)}(x)$$

and

$$C_n = (-1)^n \frac{(2n)!}{n!(n+1)!}2^{-2n}.$$

To this point, the boundary conditions are not specified and the expansion has only a formal meaning. The length parameter is introduced with the implication that it links the Compton wavelength $\ell_m = \hbar/mc$ of a particle with its mass m. The expression suggests a nonlocal nature of particles whose properties are influenced by a self-interaction without an external cause. The hitherto not specified boundary conditions raise essential difficulties—some of which are discussed in detail with their physical implications.

In Section 1 we consider scalar particles and construct Green functions and solutions. Section 2 is devoted to another representation of the Green's function and emphasis is put on stochasticity of space and time connected with a particle and plane wave solution. Quantization of nonlocal fields is the topic of Section 3. Commutation relations, Pauli–Jordan functions, and properties of Green's functions for local field equations are recalled and compared with the nonlocal field results. The static limit ($x^0 = 0$) is studied in Section 4 in which the classical Yukawa potential result is generalized for extended particles. Section 5 is devoted to an infinite-order Schrödinger equation in momentum space. In Section 6, an exact solution of the one-dimensional Schrödinger equation and its asymptotic behavior is discussed. The discrete spectrum and properties of the ground-state wave function are considered in Section 7. In Section 8, some applications to Higgs fields are given. In Sections 9 and 10 we discuss some mathematical representations and basic concepts of generalized functions. In Sections 11, 12, and 13 we study vacuum fluctuations, extra-dimensional methods in the potential theory, and electromagnetic interactions of extended electrical charges and their consequences, as physical applications of the square-root operator formalism. In Section 14 the square-root nonlocal quantum electrodynamics is constructed. Section 15 deals with concrete structures of extended charges and these are associated with nonlocal potentials and photon propagators. We also discuss here the potential of the torus and its nonlocal photon propagator as the appearance of superfields in terms of the photon and the massless fermion (photino).

1. EQUATION OF MOTION FOR SCALAR PARTICLES

1.1. Square-Root Klein–Gordon Equation

Let us consider the nonlocal field equation (7) and its Green function in momentum representation

$$\tilde{\Omega}_c(p) = -\frac{1}{\sqrt{m^2 - p^2 - i\varepsilon}} = \int_{-m}^{m} d\lambda\, \rho_m(\lambda)\tilde{S}(\lambda, \hat{p}) \qquad (8)$$

where the distribution

$$\rho_m(\lambda) = \frac{1}{\pi}(m^2 - \lambda^2)^{-\frac{1}{2}} \qquad (9)$$

has properties like

$$\int_{-m}^{m} \rho_m(\lambda)\, d\lambda = 1, \quad \int_{-m}^{m} d\lambda\, \lambda\rho_m(\lambda) = 0, \quad \int_{-m}^{m} d\lambda\, \lambda^2\rho_m(\lambda) = \frac{1}{2}m^2. \qquad (10)$$

And

$$\tilde{S}(\lambda, \hat{p}) = \frac{1}{i}\frac{\lambda + \hat{p}}{\lambda^2 - p^2 - i\varepsilon} \qquad (11)$$

is the Dirac spinor propagator with random mass λ in momentum space. In expression (8) we have used the Dirac relation $m^2 - p^2 = (m - \hat{p})(m + \hat{p})$, the relation $\hat{p} = \gamma^\nu p_\nu$, and the Feynman parametric formula

$$\frac{1}{a^{n_1}b^{n_2}} = \frac{\Gamma(n_1 + n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 dx\, x^{n_1-1}(1 - x)^{n_2-1} \frac{1}{[ax + b(1 - x)]^{n_1+n_2}}. \qquad (12)$$

In our case $n_1 = n_2 = 1/2$ and $\Gamma(1/2) = \sqrt{\pi}$. The Green's function (8) in x space takes the form

$$\Omega_c(x-y) = \int_{-m}^{m} d\lambda\, \rho_m(\lambda) S_c(x-y,\lambda), \tag{13}$$

where

$$S_c(x-y,\lambda) = \frac{1}{(2\pi)^4 i} \int d^4p\, e^{-ip(x-y)} \frac{\hat{p}+\lambda}{\lambda^2 - p^2 - i\varepsilon} \tag{14}$$

is the Dirac spinor propagator of the random mass λ.

On the other hand, the propagator of the nonlocal field $\varphi(x)$ has the representation

$$\varphi(x) = \int_{-m}^{m} d\lambda\, \rho_m(\lambda)\psi(x,\lambda), \tag{15}$$

where $\psi(x,\lambda)$ is the Dirac spinor with random mass λ, which can be obtained from the T Product

$$\begin{aligned}\Omega_c(x-y) &= \langle 0|T\{\varphi(x)\varphi^*(y)\}|0\rangle \\ &= \int_{-m}^{m}\int_{-m}^{m} d\lambda_1\, d\lambda_2\, \rho_m(\lambda_1)\rho_m(\lambda_2)\langle 0|T\{\psi(x,\lambda_1)\bar{\psi}(y,\lambda_2)\}|0\rangle.\end{aligned} \tag{16}$$

It is natural to assume that

$$\langle 0|T\{\psi(x,\lambda_1)\bar{\psi}(y,\lambda_2)\}|0\rangle = \frac{\delta(\lambda_1-\lambda_2)S_c(x-y,\lambda_1)}{\rho_m(\lambda_1)}, \tag{17}$$

where $S_c(x,\lambda)$ is given by expression (14). Thus definition (16), with the covariance property (17), yields relation (13). We derive from formulas (13)–(17) that the square-root Klein–Gordon equation (7) describes an extended field (15) with the propagator (13) and the finite mass distribution (9).

1.2. Square-Root Double Klein–Gordon Equation

Next, we consider the square-root nonlocal equation

$$\sqrt{(m^2-\Box)(m^2+\Box)}\phi(x) = 0. \tag{18}$$

This equation is invariant under the transformation $m \to im$. It relates a mathematical particle, a tachyon, and its dual physical particle partner. The Green function in momentum space is

$$\widetilde{D}_c(p) = \int_{-m^2}^{m^2} d\lambda\, \rho_{m^2}(\lambda)\widetilde{\Delta}_c\big(p,\sqrt{\lambda}\big), \tag{19}$$

where

$$\rho_{m^2}(\lambda) = \frac{1}{\pi}(m^4 - \lambda^2)^{-\frac{1}{2}}. \tag{20}$$

The propagator of the scalar particle with random mass $\sqrt{\lambda}$ is

$$\widetilde{\Delta}_c\big(p, \sqrt{\lambda}\big) = \frac{1}{i}\frac{1}{\lambda - p^2 - i\varepsilon}.$$

For Eq. (18), expressions (13), (15), and (17) take the following forms

$$D_c(x-y) = \int_{-m^2}^{m^2} d\lambda\, \rho_{m^2}(\lambda)\Delta_c\big(x-y, \sqrt{\lambda}\big), \tag{21}$$

$$\phi(x) = \int_{-m^2}^{m^2} d\lambda\, \rho_{m^2}(\lambda)\sigma_\lambda(x), \tag{22}$$

and

$$\begin{aligned} D_c(x-y) &= \langle 0|T\{\phi(x)\phi(y)\}|0\rangle \\ &= \int_{-m^2}^{m^2}\int_{-m^2}^{m^2} d\lambda_1\, d\lambda_2\, \rho_{m^2}(\lambda_1)\rho_{m^2}(\lambda_2)\langle 0|T[\sigma_{\lambda_1}(x)\sigma_{\lambda_2}(y)]|0\rangle, \end{aligned} \tag{23}$$

where $\sigma_\lambda(x)$ is the scalar local field with mass $\sqrt{\lambda}$, satisfying

$$\langle 0|T[\sigma_{\lambda_1}(x)\sigma_{\lambda_2}(y)]|0\rangle = \frac{\delta(\lambda_1-\lambda_2)\Delta_c\left(x-y, \sqrt{\lambda_1}\right)}{\rho_{m^2}(\lambda_1)} \tag{24}$$

with properties as given in (17). In expressions (21) and (24), the scalar particle propagator

$$\Delta_c\big(x, \sqrt{\lambda}\big) = \frac{1}{(2\pi)^4}\frac{1}{i}\int d^4p\, e^{-ipx}\frac{1}{\lambda - p^2 - i\varepsilon} \tag{25}$$

is used.

2. STOCHASTICITY OF SPACE–TIME

2.1. Nonlocality and Fluctuation in Space–Time Points

Here we use another (but equivalent) representation for Green functions and solutions as obtained in Section 1,

$$\Omega_c(x-y) = \int_{-1}^{1} d\lambda\, \rho(\lambda)S_c(x-y, m\lambda)$$

and (26)

$$D_c(x-y) = \int_{-1}^{1} d\lambda\, \rho(\lambda)\Delta_c\big(x-y, m\sqrt{\lambda}\big),$$

where λ plays the role of a random parameter. Solutions (15) and (22) are rewritten in the forms

$$\varphi(x) = \int_{-1}^{1} d\lambda\, \rho(\lambda)\psi(x, m\lambda)$$

and (27)

$$\phi(x) = \int_{-1}^{1} d\lambda\, \rho(\lambda)\sigma\big(x, m\sqrt{\lambda}\big).$$

It can be verified that expressions (26) and (27) can be interpreted in terms of fluctuating space–time points

$$\begin{aligned} \Omega_c(x-y) &= \int_{-1}^{1} d\lambda\, \rho(\lambda)\lambda^3 S_c(\lambda(x-y)),\\ D_c(x-y) &= \int_{-1}^{1} d\lambda\, \rho(\lambda)\lambda\Delta_c\big(\sqrt{\lambda}(x-y)\big), \end{aligned} \tag{28}$$

and

$$\begin{aligned} \varphi(x) &= \int_{-1}^{1} d\lambda\, \rho(\lambda)\lambda^{\frac{3}{2}}\psi(x\lambda),\\ \phi(x) &= \int_{-1}^{1} d\lambda\, \rho(\lambda)\lambda^{\frac{1}{2}}\sigma\big(x\sqrt{\lambda}\big). \end{aligned} \tag{29}$$

In expressions (26)–(29) we used the distribution

$$\rho(\lambda) = \frac{1}{\pi}(1-\lambda^2)^{-\frac{1}{2}}. \tag{30}$$

The space–time points $x\lambda$ and $x\sqrt{\lambda}$ have the meaning

$$\begin{aligned} x\lambda &= x_o\lambda, \mathbf{x}\lambda\\ x\sqrt{\lambda} &= x_o\sqrt{\lambda}, \mathbf{x}\sqrt{\lambda}. \end{aligned} \tag{31}$$

In Eqs. (28), the functions $S_c(\lambda x)$ and $\Delta_c(x\sqrt{\lambda})$ correspond to local propagators of spinor fields $\psi(x\lambda)$ and a scalar particle $\sigma(x\sqrt{\lambda})$ in (29) with mass m, respectively.

In the given case, T products or covariances (17) and (24) do not change

$$\langle 0|T\{\psi(x\lambda_1)\bar{\psi}(y\lambda_2)\}|0\rangle = \frac{\delta(\lambda_1-\lambda_2)S_c(\lambda_1(x-y))}{\rho(\lambda_1)}$$
$$\langle 0|T\left\{\sigma\left(x\sqrt{\lambda_1}\right)\sigma\left(y\sqrt{\lambda_2}\right)\right\}|0\rangle = \frac{\delta(\lambda_1-\lambda_2)\Delta_c\left(\sqrt{\lambda_1}(x-y)\right)}{\rho(\lambda_1)}. \tag{32}$$

A physical interpretation of expressions (28) and (29) implies that nonlocal operators induce fluctuations of particle space–time coordinates, with weight λ or $\sqrt{\lambda}$ for spin 1/2 and 0 particles respectively. The distribution $\rho(\lambda)$, defined in (30), does not depend on the spin statistics.

2.2. Plane Wave Solutions

The functions $\sigma(x\sqrt{\lambda})$ and $\psi(x\lambda)$ in (29) satisfy the Klein–Gordon and the Dirac equations. Here we study nonlocal fields for these equations. With the usual Dirac plane wave solution

$$\psi_p(x\lambda) = \frac{u(k)}{\sqrt{2k^o}}\frac{1}{(2\pi)^{\frac{3}{2}}}\, e^{-ikx\lambda}, \qquad k_o = \sqrt{m^2+\mathbf{k}^2}, \tag{33}$$

we represent $\varphi(x)$ in (29) in the form

$$\varphi_p(x) = \frac{u(k)}{\sqrt{2k^o}}\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-1}^{1} d\lambda\, \rho(\lambda)\lambda^{\frac{3}{2}}\, e^{-ikx\lambda}. \tag{34}$$

Some calculations yield an expression in terms of confluent hypergeometric functions like

$$\varphi_p(x) = \frac{u(k)}{\sqrt{2k^o}}\frac{1-i}{6}\frac{\Gamma^2\left(\frac{1}{4}\right)}{(2\pi)^2}\left\{{}_1F_2\left(\frac{5}{4};\frac{1}{2};\frac{7}{4};-\frac{(kx)^2}{4}\right)\right.$$
$$\left. +\,{}_1F_2\left(\frac{7}{4};\frac{3}{2};\frac{9}{4};-\frac{(kx)^2}{4}\right)\frac{72\pi^2}{5\Gamma^4\left(\frac{1}{4}\right)}(kx)\right\}. \tag{35}$$

The plane wave functions $\phi(x)$ in (29) for the case

$$\sigma_p\left(x\sqrt{\lambda}\right) = (2\pi)^{-\frac{3}{2}}(2k^o)^{-\frac{1}{2}}\, e^{-ikx\sqrt{\lambda}} \tag{36}$$

encounter some difficulties due to the factor $\sqrt{\lambda}$ in (36). To handle this problem, we transform the functions like

$$\phi_p(x) = N\int_{-1}^{1} d\lambda\, \rho(\lambda)\lambda^{\frac{1}{2}}\, e^{-i\Lambda\sqrt{\lambda}}$$

into the form

$$\phi_{\mathrm{p}}(x) = 2N\left[\int_0^1 du\, u^2 \rho(u^2)(ie^{\Lambda u} + e^{-i\Lambda u})\right] \tag{37}$$

with

$$N = (2\pi)^{-\frac{3}{2}}(2k^o)^{-\frac{1}{2}}, \qquad \Lambda = kx.$$

Differentiation of (37) for Λ, followed by integration by parts yields

$$\frac{\partial\phi_{\mathrm{p}}}{\partial\Lambda} = \frac{i}{\pi}N\Lambda\left[\int_0^1 du\,\sqrt{1-u^4}(e^{\Lambda u} + ie^{-i\Lambda u})\right].$$

With the expansion, whose details are found in Section 5,

$$\sqrt{1+u^2} = 1 + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(2n)!}{n!(n+1)!}\frac{u^{2n+2}}{2^{2n}} \tag{38}$$

we get

$$\begin{aligned}
\phi_{\mathrm{p}}(x) = \frac{i}{\pi}N\Bigg\{&\frac{\pi}{2}\left[Q(\Lambda) + \int d\Lambda\,(L_1(\Lambda) - L_1(-i\Lambda))\right]\\
&+ \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{n!(n+1)!}\frac{1}{2^{2n}}\Bigg[\frac{1}{2}B\left(n+\frac{3}{2},\frac{3}{2}\right)\int d\Lambda\cdot\Lambda\\
&\times\left({}_1F_2\left(n+\frac{3}{2};\frac{1}{2},n+3;\frac{\Lambda^2}{4}\right) + i\,{}_1F_2\left(n+\frac{3}{2};\frac{1}{2},n+3;-\frac{\Lambda^2}{4}\right)\right)\\
&+ \frac{1}{2}B\left(n+2,\frac{3}{2}\right)\int d\Lambda\cdot\Lambda^2\left({}_1F_2\left(n+2;\frac{3}{2},n+\frac{7}{2};\frac{\Lambda^2}{4}\right)\right.\\
&\left.+\,{}_1F_2\left(n+2;\frac{3}{2},n+\frac{7}{2};-\frac{\Lambda^2}{4}\right)\right)\Bigg]\Bigg\} + \text{const.},
\end{aligned} \tag{39}$$

where $Q(\Lambda) = iI_o(-i\Lambda) - I_o(\Lambda)$ and $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$; $L_1(x)$ and $I_o(x)$ are modified Struve and modified Bessel functions, respectively.

3. QUANTIZATION OF SQUARE-ROOT THEORY

3.1. Commutation Relations of Nonlocal Fields

The free fields $\varphi(x)$ and $\phi(x)$, which obey Eq. (7) and (18), become operator-valued and contain positive and negative frequency parts $\varphi(x) = \varphi^+(x) + \varphi^-(x)$ and $\phi(x) = \phi^+(x) + \phi^-(x)$. In this section we investigate their commutation relations and Green functions.

Representation (27) is convenient when we work with quantized fields $\varphi(x)$ and $\phi(x)$. We consider here only scalar particles. First we notice that definitions

(16) and (23) for T-product operators, as well as properties (17) and (24), remain valid. The usual product of quantized fields $\phi(x)$ in (27), $\sigma(x, m\sqrt{\lambda}) = \sigma(x, \lambda)$ combined with the stochastic average, or expectation value, of σ is

$$[\phi(x)\phi(y)]_{\mathrm{E}} = \int_{-1}^{1}\int_{-1}^{1} d\lambda_1\, d\lambda_2\, \rho(\lambda_1)\rho(\lambda_2)\, \mathrm{Expec}\{\sigma(x, \lambda_1)\sigma(y, \lambda_2)\}, \qquad (40)$$

where

$$\mathrm{Expec}\{\sigma(x, \lambda_1)\sigma(y, \lambda_2)\} = \frac{\delta(\lambda_1 - \lambda_2)\sigma(x, \lambda_1)\sigma(y, \lambda_2)}{\rho(\lambda_1)} \qquad (41)$$

has the form given in (24). The commutator of field operators $\phi(x)$ takes the form

$$[\phi(x), \phi(y)]_{-} = \int_{-1}^{1}\int_{-1}^{1} d\lambda_1\, d\lambda_2\; \rho(\lambda_1)\rho(\lambda_2)\, \mathrm{Expec}[\sigma(x, \lambda_1), \sigma(y, \lambda_2)]. \qquad (42)$$

Making use of (41) one gets

$$\begin{aligned} D(x - y) = D^{+}(x - y) + D^{-}(x - y) &= [\phi(x), \phi(y)]_{-} \\ &= \int_{-1}^{1} d\lambda\, \rho(\lambda)\Delta(x - y, \lambda), \end{aligned} \qquad (43)$$

where $\Delta(x - y, \lambda)$ is the Pauli–Jordan function of the scalar particle, with a mass $m\sqrt{\lambda}$ (Bogolubov and Shirkov, 1980), and

$$\Delta(x, \lambda) = \frac{1}{2\pi i}\varepsilon(x^o)\delta(\chi) - \frac{m\sqrt{\lambda}}{4\pi i\sqrt{\chi}}\theta(\chi)\varepsilon(x^o)J_1\big(m\sqrt{\lambda\chi}\big), \quad \chi = (x^o)^2 - \mathbf{x}^2. \qquad (44)$$

3.2. Green Functions of Nonlocal Operators

We use the Green function of the field $\phi(x)$ in (26)

$$D_{\mathrm{c}}(x - y) = \int_{-1}^{1} d\lambda\, \rho(\lambda)\Delta_{\mathrm{c}}(x - y, \lambda). \qquad (45)$$

Here

$$\begin{aligned} \Delta_{\mathrm{c}}(x, \lambda) = {} & \frac{1}{4\pi i}\delta(\chi) - \frac{m\sqrt{\lambda}}{8\pi i\sqrt{\chi}}\theta(\chi)\big[J_1\big(m\sqrt{\lambda\chi}\big) - iN_1\big(m\sqrt{\lambda\chi}\big)\big] \\ & + \frac{m\sqrt{\lambda}}{4\pi^2\sqrt{-\chi}}\theta(-\chi)K_1\big(m\sqrt{-\lambda\chi}\big). \end{aligned} \qquad (46)$$

Retarded and advanced propagators for square-root scalar particle are defined as

$$D^{\mathrm{adv}}(x - y) = \int_{-1}^{1} d\lambda\, \rho(\lambda)\Delta^{\mathrm{adv}}(x - y, \lambda)$$

and

$$D^{\text{ret}}(x-y) = \int_{-1}^{1} d\lambda\, \rho(\lambda)\Delta^{\text{ret}}(x-y,\lambda). \tag{47}$$

Here

$$\Delta^{\text{adv}}(x) = \Delta^{\text{ret}}(-x) = \frac{1}{2\pi i}\theta(-x^o)\left\{\delta(\chi) - \frac{m\sqrt{\lambda}}{2\sqrt{\chi}}\theta(\chi)J_1\left(m\sqrt{\lambda\chi}\right)\right\}. \tag{48}$$

Finally, definition of $D^{\pm}(x)$ in (43) is

$$D^{\pm}(x) = \int_{-1}^{1} d\lambda\ \rho(\lambda)\Delta^{\pm}(x,\lambda). \tag{49}$$

Next, we study these functions in the light cone neighborhood to verify properties of singularities. Using a decomposition of the cylindrical Bessel functions in (44), (46), and (48) one finds

$$\begin{aligned}
\Delta(x,\lambda) &= \frac{1}{2\pi i}\varepsilon(x^o)\delta(\chi) - \frac{m^2\lambda}{8\pi i}\varepsilon(x^o)\theta(\chi) + O(\chi),\\
\Delta_{\text{c}}(x,\lambda) &= \frac{1}{4\pi i}\delta(\chi) - \frac{1}{4\pi^2}\frac{1}{\chi} - \frac{m^2\lambda}{16\pi i}\theta(\chi) + \frac{m^2\lambda}{8\pi^2}\log\frac{m\sqrt{\lambda}|\chi|^{\frac{1}{2}}}{2}\\
&\quad + O\left(\sqrt{|\chi|}\log|\chi|\right), \qquad (50)\\
\Delta^{\text{ret}}(x,\lambda) &= \frac{1}{2\pi i}\theta(x^o)\delta(\chi) - \frac{m^2\lambda}{8\pi i}\theta(x)\theta(\chi) + O(\chi),
\end{aligned}$$

and

$$\Delta^{\pm}(x,\lambda) = \frac{1}{4\pi i}\varepsilon(x^o)\delta(\chi) \pm \frac{1}{4\pi^2\chi} \mp \frac{m^2\lambda}{8\pi^2}\log\frac{m\sqrt{\lambda}|\chi|^{\frac{1}{2}}}{2} - \frac{m^2\lambda}{16\pi i}\varepsilon(x^o)\theta(\chi).$$

It is verified that singularities associated with the mass terms disappear after integration over λ in (43), (45), (47), and (49). Remaining singularities are connected with space–time properties only, and the removal of these singularities requires a careful treatment of the space–time structure at short distances. This problem is beyond the scope of this work.

3.3. Causality

One of the principle requirements of quantum field theory is the satisfaction of causality (Bogolubov and Shirkov, 1980). Generally, there are two types of causality conditions to consider.

1. Microcausality is manifest as a requirement imposed on the Heisenberg fields $\sigma(x)$, which must be locally commutable

$$[\sigma(x), \sigma(y)]_- = 0$$

in a spacelike region $x \sim y$. As microcausality condition for the S matrix

$$\frac{\delta}{\delta\sigma(x)}\left(\frac{\delta S}{\delta\sigma(y)} S^+\right) = 0 \quad \text{for} \quad x \leq y.$$

2. Macrocausality is equivalent to the requirement that "effective wave packet"

$$\phi(x) = \int d^4y\, K(x-y)\sigma(y)$$

fall off rapidly when the function $\sigma(x)$ describing the "initial wave packet" falls off rapidly outside region G_σ (Efimov, 1977; Namsrai, 1986).

It is surprising that the theory, based upon the square-root Klein–Gordon operator, has microcausal validity. This follows directly from the representation (43) with (44). The usual local Pauli–Jordan function (44) vanishes outside the light cone $\chi = x_o^2 - \mathbf{x}^2 < 0$ for any mass $m\sqrt{\lambda}$. As a consequence, all (anti)commutators of the nonlocal field operators (27) tend to zero when their arguments are separated by a spacelike interval, all in accordance with the formula (43). Its physical consequence is that events are independent if they are separated by spacelike intervals.

4. NEW POTENTIALS

Photon exchange generates the electromagnetic interaction whereas boson exchanges are responsible for the strong short-range nuclear forces. In particular, the Coulomb and Yukawa potentials are related to single photon and single pion propagators in the static limit

$$U_{\mathrm{c}}(r) = \frac{e}{(2\pi)^3}\int d\mathbf{p}\, e^{i\mathbf{pr}}\frac{1}{\mathbf{p}^2} = \frac{e}{4\pi r}, \tag{51}$$

$$U_{\mathrm{Y}}(r) = \frac{g}{(2\pi)^3}\int d\mathbf{p}\, e^{i\mathbf{pr}}\frac{1}{m^2+\mathbf{p}^2} = \frac{g}{4\pi r}e^{-mr}. \tag{52}$$

In analogy, if there exist forces due to exchange of nonlocal field particles, from Eqs. (7) and (18) follows

$$U_1(r) = \frac{\left(\frac{g_1}{m}\right)}{(2\pi)^3}\int d\mathbf{p}\, e^{i\mathbf{pr}}\frac{1}{\sqrt{m^2+\mathbf{p}^2}}, \tag{53}$$

$$U_2(r) = \frac{g_2}{(2\pi)^3} \int_{-1}^{1} d\lambda\, \rho(\lambda)\sqrt{\lambda} \int d\mathbf{p}\, e^{i\mathbf{pr}\sqrt{\lambda}} \frac{1}{m^2 + \mathbf{p}^2}. \tag{54}$$

In the latter, we used representation (28) in R_3 space

$$D_c(\mathbf{x}) = \int_{-1}^{1} d\lambda \frac{1}{\lambda} \rho(\lambda) \Delta_c\left(\mathbf{x}\sqrt{\lambda}\right), \tag{55}$$

where

$$\Delta_c\left(\mathbf{x}\sqrt{\lambda}\right) = \frac{\lambda\sqrt{\lambda}}{(2\pi)^3} \int d\mathbf{p} \frac{1}{m^2 + \mathbf{p}^2} e^{i\mathbf{px}\sqrt{\lambda}}. \tag{56}$$

After some calculations, we find

$$U_1(r) = \frac{1}{2\pi^2 r} g_1 K_1(mr) \tag{57}$$

and

$$U_2(r) = m g_2 f(x), \quad x = mr. \tag{58}$$

Here

$$f(x) = \frac{1}{4\pi^2 x} \int_0^1 \frac{d\lambda}{\sqrt{1-\lambda^2}} \left(e^{-x\sqrt{\lambda}} + \cos x\sqrt{\lambda}\right). \tag{59}$$

Asymptotically, the behavior of (57) is given by

$$U_1(r) = \begin{cases} \dfrac{1}{2\pi^2 r^2} \dfrac{g_1}{m} & r \to 0 \\ \dfrac{1}{(2\pi r)^{\frac{3}{2}}} \dfrac{1}{\sqrt{m}} e^{-mr} & r \to \infty. \end{cases} \tag{60}$$

The function (59) has a series representation in x like

$$f(x) = \frac{1}{8\pi x\sqrt{\pi}} \{-F_1(x) + F_2(x)\}, \tag{61}$$

where

$$F_1(x) = \sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{2n-1}{4} + \frac{1}{2}\right)}{\Gamma\left(\frac{2n-1}{4} + 1\right)} \frac{x^{2n-1}}{(2n-1)!}$$

and

$$F_2(x) = \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1+n}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)} (1 + (-1)^n) \frac{x^{2n}}{(2n)!}.$$

Notice, the equivalent local potential $U_2(r)$ (58) is an oscillatory function and, different from commonly used potentials, is periodically attractive and repulsive (see Fig. 1).

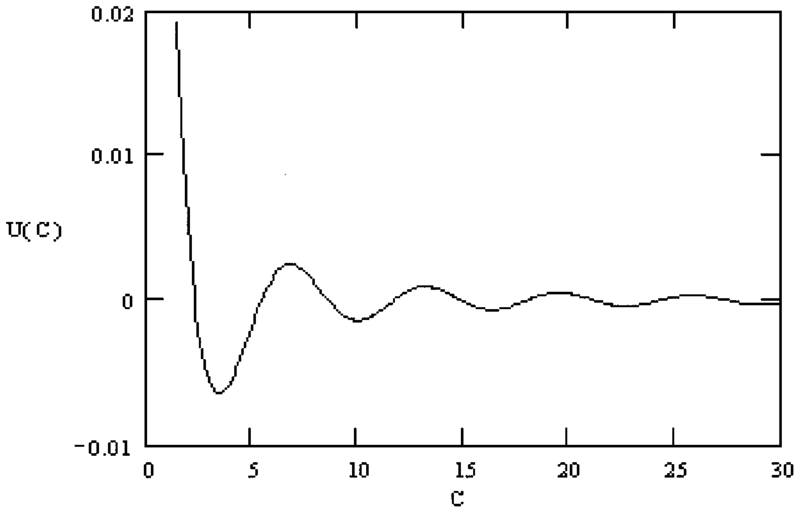

Fig. 1. Behavior of the potential $U_2(c) = f(c)$, $(c = mr)$.

5. SCHRÖDINGER EQUATION WITH NONLOCAL POTENTIAL

The usual Schrödinger equation is an approximation with respect to a nonlocal potential case

$$i\frac{\partial\psi}{\partial t} = \sqrt{m^2 - \nabla^2}\psi + U\psi, \tag{62}$$

where $U(r)$ is a local potential, like the point charge Coulomb potential

$$U = -\frac{e^2}{r}. \tag{63}$$

Equation (62), with generalized potentials, can be written with a series representation like

$$\sqrt{m^2 - \nabla^2} = m + \sqrt{-\nabla^2}\int_0^\infty \frac{d\lambda}{\lambda} e^{-m\lambda} J_1\big(\lambda\sqrt{-\nabla^2}\big). \tag{64}$$

Making use of the series

$$J_1\big(\lambda\sqrt{-\nabla^2}\big) = \frac{\lambda\sqrt{-\nabla^2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}\left(-\frac{\lambda^2\nabla^2}{4}\right)^n,$$

we get an infinite order differential equation

$$i\frac{\partial\psi}{\partial t} = (m+U)\psi + \frac{m}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{n!(n+1)!}2^{-2n}\big(-\ell_m^2\nabla^2\big)^{n+1}\psi, \tag{65}$$

where $\ell_m = \hbar/mc$ is the Compton wavelength of the particle with mass m.

It should be noted that the plane wave $\psi_{\mathrm{p}} \sim e^{-iEt+i\mathbf{px}}$, $E = \sqrt{m^2 + \mathbf{p}^2}$ satisfies (62) with $U = 0$. For the stationary case

$$\psi = e^{-iEt}\psi(\mathbf{x}), \tag{66}$$

Eq. (65) assumes the form

$$(E - m - U)\psi(\mathbf{x}) = \int d\mathbf{y}\; K_{\ell_m}(\mathbf{x} - \mathbf{y})\psi(\mathbf{y}), \tag{67}$$

where

$$K_{\ell_m}(\mathbf{x} - \mathbf{y}) = K_{\ell_m}\left(-\ell_m^2 \nabla^2\right)\delta^{(3)}(x - y)$$

is a generalized function. Here

$$K_{\ell_m}\left(-\nabla^2 \ell_m^2\right) = \frac{m}{2}\sum_{n=0}^{\infty} C_n\left(-\nabla^2 \ell_m^2\right)^{n+1} \tag{68}$$

with $C_n = (-1)^n(2n)!2^{-2n}/[n!(n+1)!]$.

Fourier transformation of Eq. (67), with the potential (63), yields its form in momentum space. In this representation $\hat{\mathbf{p}}$ and $\hat{\mathbf{x}}$ are

$$\hat{\mathbf{p}}\psi_{\mathbf{p}} = \mathbf{p}\psi_{\mathbf{p}}, \hat{\mathbf{x}}\psi_{\mathbf{p}} = i\frac{\partial}{\partial \mathbf{p}}\psi_{\mathbf{p}},$$

where $\psi_{\mathbf{p}}$ is the wave function in momentum representation. The operator $\hat{r}^{-1}$ is not well defined. However, its ambiguity can be removed by adding suitable terms to the potential that are proportional to a δ function and its derivatives at the origin. Similarly, we can multiply the equation by the operator $\hat{r}$ from the left. Thus we get ($\psi_{\mathbf{p}} = \tilde{\psi}(\mathbf{p})$)

$$\sqrt{\left(i\frac{d}{dp_x}\right)^2 + \left(i\frac{d}{dp_y}\right)^2 + \left(i\frac{d}{dp_z}\right)^2}\left\{\left[\sqrt{m^2 + \mathbf{p}^2} - E\right]\tilde{\psi}(\mathbf{p})\right\} = e^2\tilde{\psi}(\mathbf{p}) \tag{69}$$

6. SCHRÖDINGER EQUATION ON THE LINE

In one dimension, Eqs. (67) and (69) are simplified like

$$(E - m - U)\psi(x) = -\frac{1}{2\pi}\int_0^{\infty} d\lambda\, e^{-m\lambda}\int_{-1}^{1} dt(1 - t^2)^{\frac{1}{2}}\frac{d^2}{dx^2} \times [\psi(x + \lambda t) + \psi(x - \lambda t)] \tag{70}$$

and

$$i\frac{d}{dp}\left\{\left[\sqrt{m^2 + p^2} - E\right]\tilde{\psi}(p)\right\} = e^2\tilde{\psi}(p). \tag{71}$$

Let us investigate Eq. (71) in detail. Differentiation of (71) gives

$$\frac{d}{dp}\log\tilde{\psi} = \left[-ie^2 - \frac{p}{\sqrt{m^2+p^2}}\right]\left[\sqrt{m^2+p^2} - E\right]^{-1} \tag{72}$$

Further, changing variable $p = m\sinh t$ and carrying out integration, one gets

$$\log c\psi = \frac{1}{i}e^2 \int \frac{dt\ \cosh t}{\cosh t - \frac{E}{m}} - \int \frac{dt\ \sinh t}{\cosh t - \frac{E}{m}} \tag{73}$$

Here c is a integration constant and we distinguish the two cases as

$$\begin{aligned} &\text{i.}\ \ E/m < 1 \\ &\text{ii.}\ \ E/m \geq 1. \end{aligned} \tag{74}$$

Then, the first case gives

$$\begin{aligned} \psi_{\mathrm{d}}(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dp\,\tilde{\psi}(p)\,e^{ipx} = \frac{\text{const}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dp\,e^{ipx}\left(\frac{p}{m}+Q\right)^{-ie^2}\frac{1}{Q-\frac{E}{m}} \\ &\quad\times\left\{\theta(-p)\ \exp\left[-ie^2\frac{E}{m}\frac{1}{\sqrt{1-\frac{E^2}{m^2}}}\ \arcsin\left(\frac{1-\frac{E}{m}Q}{Q-\frac{E}{m}}\right)\right]\right. \\ &\quad\left.+\,\theta(p)\ \exp\left[ie^2\frac{E}{m}\frac{1}{\sqrt{1-\frac{E^2}{m^2}}}\ \arcsin\left(\frac{1-\frac{E}{m}Q}{Q-\frac{E}{m}}\right)\right]\right\}, \end{aligned} \tag{75}$$

where $Q = \sqrt{1+p^2/m^2}$. After some transformations we obtain an equivalent form

$$\begin{aligned} \psi_{\mathrm{d}}(x) &= \frac{\text{const}}{\sqrt{2\pi}}\int_0^{\infty}\frac{dp}{Q-\frac{E}{m}}\left\{e^{ipx}\left(\frac{p}{m}+Q\right)^{-ie^2}\left[\frac{i(1-\frac{E}{m}Q)+\sqrt{\frac{p^2}{m^2}}\gamma}{Q-\frac{E}{m}}\right]^{\frac{E}{m}\frac{e^2}{\gamma}}\right. \\ &\quad\left.+\,e^{-ipx}\left(Q-\frac{p}{m}\right)^{-ie^2}\left[\frac{i(1-\frac{E}{m}Q)+\sqrt{\frac{p^2}{m^2}}\gamma}{Q-\frac{E}{m}}\right]^{-\frac{E}{m}\frac{e^2}{\gamma}}\right\}. \end{aligned} \tag{76}$$

Here $\gamma = \sqrt{1-E^2/m^2}$. For the second case in (74) we have

$$\psi_{\mathrm{c}}(x) = \frac{\text{const}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dp\,e^{ipx}\left(\frac{p}{m}+Q\right)^{-ie^2}\frac{1}{Q-\frac{E}{m}}\left[\frac{1-\frac{E}{m}+\gamma'\frac{m}{p}(Q-1)}{1-\frac{E}{m}-\gamma'\frac{m}{p}(Q-1)}\right]^{\eta}, \tag{77}$$

where $\gamma' = \sqrt{E^2/m^2-1}$ and $\eta = -i(E/m)(e^2/\gamma')$.

The expressions (75) and (77) describe wave functions of discrete and continuous spectra. The saddle point method gives the asymptotic behaviors of the solution (75) at $x \to \pm\infty$

$$\psi_{\rm d}(x) \sim \text{const}\,\frac{m}{e}\frac{\sqrt{2\pi E}}{(m^2-E^2)^{\frac14}}e^{-m\gamma x}\left(\frac{E}{m}+i\gamma\right)^{-ie^2}\left\{\left[2i\frac{mx}{e^2}\gamma^2\right]^{\frac{E}{m}\frac{e^2}{\gamma}} + \left[2i\frac{mx}{e^2}\gamma^2\right]^{-\frac{E}{m}\frac{e^2}{\gamma}}\right\} + \text{const}\left(-\frac{m}{e}i\right)\sqrt{\frac{2\pi}{m}}e^2x^{-\frac32}\,e^{-mx}(i)^{-ie^2} \times\left\{\left[-i\frac{m}{E}(1+\gamma)\right]^{\frac{E}{m}\frac{e^2}{\gamma}} + \left[-i\frac{E}{m}(1+\gamma)\right]^{-\frac{E}{m}\frac{e^2}{\gamma}}\right\} \tag{78}$$

for $x \to +\infty$ and

$$\psi_{\rm d}(x) \sim \text{const}\,\frac{m}{e}i\frac{\sqrt{2\pi E}}{(m^2-E^2)^{\frac14}}\left(\frac{E}{m}-i\gamma\right)^{-ie^2}e^{m\gamma x}\left\{\left[2i\frac{mx}{e^2}\gamma^2\right]^{\frac{E}{m}\frac{e^2}{\gamma}} + \left[2i\frac{mx}{e^2}\gamma^2\right]^{-\frac{E}{m}\frac{e^2}{\gamma}}\right\} + \text{const}\left(-\frac{m}{E}\right)\sqrt{\frac{2\pi}{m}}x^{-\frac32}\,e^2(-i)^{-ie^2}\,e^{mx} \times\left\{\left[-\frac{m}{E}i(1+\gamma)\right]^{\frac{E}{m}\frac{e^2}{\gamma}} + \left[-\frac{m}{E}i(1+\gamma)\right]^{-\frac{E}{m}\frac{e^2}{\gamma}}\right\} \tag{79}$$

for $x \to -\infty$. These asymptotic boundary conditions determine the physical wave functions, i.e., they decrease rapidly at infinity.

7. DISCRETE SPECTRUM AND GROUND-STATE WAVE FUNCTION

The fundamental solution (75), with boundary conditions (78) and (79), shows that wave function of the Schrödinger equation (70) is an analytic function only if

$$\frac{E}{m}\frac{e^2}{\sqrt{1-\frac{E^2}{m^2}}} = n, \quad n = 1, 2, \ldots \tag{80}$$

They yield the discrete spectrum, or bound-state energies

$$E = -\frac{m}{\sqrt{1+\frac{e^4}{n^2}}}. \tag{81}$$

For weak coupling (e^2 is small), we obtain Bohr's results for hydrogen-like atoms ($e^2 \to Ze^2$). With the nonlocal Schrödinger equation, we determine the spectrum from the analytic properties of the solutions. On the other hand, in the strong coupling limit $e^2 \to g^2 \gg 1$ the discrete energy spectrum is $E = -mn/g^2$ for some fixed $n_{\rm f} \leq g^2$ where $n \leq n_{\rm f}$.

The ground-state wave function, with $n = 1$ in the nonlocal Schrödinger equation solution, is

$$\psi_1(x) = \frac{\rm const}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp\, e^{ipx} \left(\frac{p}{m} + Q\right)^{-ie^2} \frac{\sqrt{\frac{p^2}{m^2}}\gamma}{\left(-\frac{E}{m} + Q\right)^2} + g(x), \tag{82}$$

where

$$g(x) = \frac{\rm const}{\sqrt{2\pi}} i \int_0^{\infty} dp \frac{\left(1 - \frac{E}{m}Q\right)}{\left(-\frac{E}{m} + Q\right)^2} \left[e^{ipx} \left(\frac{p}{m} + Q\right)^{-ie^2} - e^{-ipx} \left(-\frac{p}{m} + Q\right)^{-ie^2} \right]. \tag{83}$$

The integrals are evaluated by contour integration and Cauchy's formula

$$\psi_1(x) = \begin{cases} \sqrt{2\pi} i \text{ const } \cdot E^2 e^{-mx\gamma} \left(\frac{E}{m} + i\gamma\right)^{-ie^2} N(x) + g(x) & \text{for } x > 0; \\ -\sqrt{2\pi} i \text{ const } \cdot E^2 e^{mx\gamma} \left(\frac{E}{m} - i\gamma\right)^{-ie^2} N(x) + g(x) & \text{for } x < 0. \end{cases}$$

Here $g(x)$ is given by (83) and $N(x) = (x - \frac{1}{E}e^2 + \frac{m}{E^2}\gamma)$. Finally, the free motion wave functions ($e^2 = 0$) and continuous spectrum for the nonlocal Schrödinger equation (70) is discussed. More details are given in Section 9.3. Standing wave Coulomb solutions are

$$\psi_{\rm f}(x) = -\text{const} \cdot 2\sqrt{2\pi} m E \frac{\sin kx}{k} \tag{84}$$

and

$$\psi_{\rm c}(x) = \frac{\rm const}{\sqrt{2\pi}} m^2 (-1)^{ie^2 \frac{E}{k}} \int_{-\infty}^{\infty} dp\, e^{ipx} \left(\frac{p}{m} + Q\right)^{-ie^2} \frac{\frac{E}{m} + Q}{(p + k)} \times (p - k)^{-1 - ie^2 \frac{E}{k}} \left[\frac{p}{m} E + m\gamma' Q\right]^{ie^2 \frac{E}{k}}, \tag{85}$$

where $k = \sqrt{E^2 - m^2}$. We see that the wave function of the free motion is the standing plane wave with the wave number k, and the wave function of the continuous spectra has an essential singular branch point at $p = k$. The infinite degeneracy of all positive-energy levels is the result of the infinite order of the nonlocal Schrödinger equation.

8. HIGGS FIELDS

Today, a Higgs particle, even if it has not been verified experimentally, is part of the standard model of electro-weak interactions and is supposed to explain the mass of particles by a spontaneous symmetry-breaking mechanism. The direct search at the LEP for the Higgs boson excludes Higgs masses below ~95.2 GeV (European Physical Society [EPS], 2000; Janot, 1997). Unitarity of the scattering amplitude requires a cutoff Λ (Altarelli and Isidori, 1994), which imposes an upper bound on the Higgs mass. For the minimal value $\Lambda \sim 1$ TeV, the upper bound on the Higgs mass (m_{H}) is $\leq$700 GeV, for $\Lambda \sim M_{\mathrm{GUT}}$, which is $\sim 10^{16}$ GeV, the Higgs mass has to be smaller than ~200 GeV (see also Namsrai, 1996a,b). In the last years, minimal supersymmetric extensions of this problem were also extensively discussed (Haber *et al.*, 1997). We have already examined in some detail implications of the nonlocality with the Klein–Gordon equation. There, it turns out that square-root operators have a physical meaning in terms of self interactions and some exact solutions are known. Alternatively, the physics generated by square-root operators may also be interpreted in terms of stochastic space–time motion. Equations (7) and (18) describe spinor and scalar fields with random mass distributions. In the massless case, Eqs. (7) and (18) describe neutrinos and photons, and dipole potentials ($U_{\mathrm{d}} \sim 1/r^2$) or Coulomb potentials ($U_{\mathrm{c}} \sim 1/r$) enter. However, we notice that a representation of square-root operators, using nonlocal generalized function $K_{\ell_m}(x)$ in (68), works only well for superheavy particles like t quarks and Higgs particles, since in this case the parameter $\ell = \hbar/mc$ becomes small. While, for massive neutrinos, axions, and other very light supersymmetric particles we must use the full square-root-operator forms (7) and (18). In another remark, concerning applications of our scheme examined in Higgs fields, we propose that motion of Higgs particles and their interaction with other particles may be given by the square-root Klein–Gordon-type equations (7), (18), or (67) with potentials (57) and (58). Why Higgs particles do not show up experimentally may have its reason in a random mass distribution

$$\rho_m(x) = \left(\frac{1}{\pi}\right)(m^4 - x^2)^{-\frac{1}{2}}$$

with the density

$$\int_{-m^2}^{m^2} dx\, \rho_m(x) = 1, \quad \int_{-m^2}^{m^2} dx\, x\rho_m(x) = 0, \quad \int_{-m^2}^{m^2} dx\, x^2\rho_m(x) = \frac{1}{2}m^2.$$

The energy of the Higgs particle is given by

$$E = \int_{-1}^{1} d\lambda\, \rho(\lambda)\sqrt{\mathbf{p}^2 + m^2\lambda^2} = \frac{2}{\pi}\sqrt{m^2 + \mathbf{p}^2}\, E\left(\frac{\pi}{2}, \frac{m}{\sqrt{m^2 + \mathbf{p}^2}}\right),$$

where $E(\pi/2, x)$ in the integrand is the complete elliptic integral of the second kind.

9. MATHEMATICAL TOOLS

9.1. Product of Nonlocal Field Operators and Green Functions

Equalities (17) and (24) allow us to get a more exact definition of nonlocal field operator products, like

$$\phi(x) \cdot \phi(y) = \int_{-1}^{1}\int_{-1}^{1} d\lambda_1\, d\lambda_2\, \rho(\lambda_1)\rho(\lambda_2)\, \text{Expec}[\sigma(x, \lambda_1)\sigma(y, \lambda_2)],$$

where, by definition,

$$\text{Expec}[\sigma(x, \lambda_1)\sigma(y, \lambda_2)] = \sigma(x, \lambda_1)\sigma(y, \lambda_2)\delta(\lambda_1 - \lambda_2)/\rho(\lambda_1). \qquad (86)$$

Thus

$$\phi(x) \cdot \phi(y) = \int_{-1}^{1} d\lambda\, \rho(\lambda)\sigma(x, \lambda)\sigma(y, \lambda). \qquad (87)$$

Definitions (86) and (87) imply that $\sigma(x, \lambda)$ is a random distribution, like a Gaussian with the variable λ representing its width, whose expectation value or covariance is given by (86). In analogy, the commutator of field operators is defined as

$$[\phi(x), \phi(y)]_- = \int_{-1}^{1} d\lambda\, \rho(\lambda)[\sigma(x, \lambda), \sigma(y, \lambda)]_-. \qquad (88)$$

Definitions (86), (87), and (88) can be generalized and used to form the product of Green functions

$$D_c(x) \cdot D_c(y) = \int_{-1}^{1} d\lambda\, \rho(\lambda)\Delta_c(x, \lambda)\Delta_c(y, \lambda). \qquad (89)$$

The definition (89) also ensures gauge invariance (see Section 14 and Namsrai, 1998).

9.2. Field Theory on the Unit Circle

Matrix elements of the S matrix are defined as products $\prod_{i\neq j} D(x_i - x_j)$, where $D(x)$ is the nonlocal propagators (26) or (28). For both scalar and spinor fields, nonlocal Green functions (28) can be defined in the complex plane on the unit circle

$$\Omega_c(x) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} \left(\frac{\ln z}{2\pi i}\right)^3 \rho\left(\frac{\ln z}{2\pi i}\right) \left[S_c\left(\frac{\ln z}{2\pi i} x\right) - S_c\left(-x \frac{\ln z}{2\pi i}\right)\right] \quad (90)$$

and

$$D_c(x) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} \left(\frac{\ln z}{2\pi i}\right) \rho\left(\frac{\ln z}{2\pi i}\right) \left[\Delta_c\left(x\sqrt{\frac{\ln z}{2\pi i}}\right) - \Delta_c\left(ix\sqrt{\frac{\ln z}{2\pi i}}\right)\right]. \quad (91)$$

Equations (90) and (91) imply a representation of all physical quantities on the unit circle, in both x or p spaces. An analogous representation exists for Green functions (26).

9.3. Boundary Conditions for The Schrödinger Equation

Let us study the solution of the one-dimensional Schrödinger equation (70) ($U = 0$), which is given by (77) with $e = 0$:

$$\psi_f(x) = \frac{\text{const}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp\, e^{ipx} \frac{1}{-\frac{E}{m} + \sqrt{1 + \frac{p^2}{m^2}}} \quad (92)$$

Integration of this expressions over the variable p encounters two poles $p_{1,2} = \pm\sqrt{E^2 - m^2} = \pm k$. The choice of circumvention for these poles dictates by boundary conditions. We distinguish four cases:

1. $\psi_f(x) = 0 \quad \text{for} \quad x < 0$
2. $\psi_f(x) = 0 \quad \text{for} \quad x > 0$
3. A particle moving to the left: $\psi_f(x) \sim e^{-ikx}, \quad x > 0$
4. A particle moving to the right: $\psi_f(x) \sim e^{ikx}, \quad x < 0.$

Thus, the first and the second cases lead to the solutions

$$\psi_{1f}(x) = -\text{const} \cdot 2\sqrt{2\pi} mE \frac{\sin kx}{k} \theta(x) \quad (93)$$

and

$$\psi_{2f}(x) = \text{const} \cdot 2\sqrt{2\pi} mE \frac{\sin kx}{k} \theta(-x). \quad (94)$$

While from the last two cases we get

$$\psi_f(x) = -\text{const} \cdot \sqrt{2\pi} \cdot imE[e^{-ikx}\theta(x) + e^{ikx}\theta(-x)]\frac{1}{k}. \quad (95)$$

It seems that the appropriate boundary conditions at the positive or negative semi axis have been written to ensure the Hamiltonian of the one-dimensional quantum system is a self-adjoint operator either on the positive or on the negative semi axis, but not on the whole axis as in a usual quantum mechanical case.

10. GENERALIZED FUNCTIONS AND THE KÄLLEN–LEHMANN REPRESENTATION

10.1. Basic Concept of Generalized Functions

Generalized functions enable to define as a density a material point or a point charge. Here we give some basic definitions of test functions and generalized functions or distributions. We shall explain their main differences in the cases of local and nonlocal quantum field theories (for further reading see Bogolubov *et al.*, 1975; Dautray and Lions, 1985; Gel'fand and Shilov, 1964/1968; Messian, 1961; Namsrai, 1986; Schwartz, 1950, 1951; Vladimirov, 1979). To describe the concept of generalized functions let us define the density generated by a material point with the mass $m = 1$ and a uniform distribution within a sphere of radius ε. The centre is at the origin 0. The average density $\rho_\varepsilon(x)$ is

$$\rho_\varepsilon(x) = \begin{cases} (4\pi\varepsilon^3/3) & \text{if} \quad |x| < \varepsilon; \\ 0 & \text{if} \quad |x| > \varepsilon. \end{cases}$$

We are interested in the density distribution when $\varepsilon \to +0$, which defines

$$\delta(x) = \begin{cases} +\infty & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x \neq 0 \end{cases} \tag{96}$$

as limit of a sequence of densities $\rho_\varepsilon(x)$ with

$$\lim_{\varepsilon\to 0} \int d^3x\, \rho_\varepsilon(x) = \int_{-\infty}^{+\infty} d^3x\, \delta(x) = 1.$$

For a continuous function $f(x)$,

$$\lim_{\varepsilon\to +0} \int d^3x\, \rho_\varepsilon(x) f(x) = f(0). \tag{97}$$

This formula denotes the weak limit of the sequence $\lim_{\varepsilon\to 0} \rho_\varepsilon(x)$ as the functional $f(0)$, which is not the function itself. It assigns to each continuous function $f(x)$ the functional value $f(0)$. This limiting process is abbreviated by the improper function $\delta(x)$ for which $\int d^3x\, \delta(x) f(x) = f(0)$. In literature one finds other brief notations like

$$(\delta, f) = f(0). \tag{98}$$

We see that $\delta(x)$ generates a point mass at the origin and

$$\int d^3x\, \delta(x) = (\delta, 1) = 1.$$

Before defining the functionals we consider the spaces of test functions. It is said that continuous functions are the test functions for the δ functional. This point of view takes as its basis of the definition of any generalized functions as a linear continuous functional onto sets of sufficiently well-defined so-called test functions. In general, test functions set up linear normalized space $\mathcal{U}$, in which the commutative and associative addition is defined as

1. $u + v = v + u$
2. $u + (v + w) = (u + v) + w$

And also

$$u + 0 = u, \qquad u + (-u) = 0$$

for each $u, v, w \in \mathcal{U}$ and $0 \in \mathcal{U}$. Moreover,

$$1 \cdot u = u, \qquad \lambda(\mu u) = (\lambda\mu)u$$

and

$$(\lambda + \mu)u = \lambda u + \mu u, \qquad \lambda(u + v) = \lambda u + \lambda v$$

for any $u, v \in \mathcal{U}$ and some (real and complex) numbers λ and μ. The real function $p(u) = \|u\|$ defined on $\mathcal{U}$ is called the norm if the following conditions are fulfilled:

(i) for any number λ, $p(\lambda u) = |\lambda| p(u)$
(ii) $p(u + v) \le p(u) + p(v)$ (triangle inequality)
(iii) if $p(u) = 0$, then $u = 0$.

From conditions (i) and (ii), it follows nonnegativity of the norm

$$0 = p(u - u) \le p(u) + p(-u) = 2p(u).$$

The functions satisfying only conditions (i) and (ii) are called the semi-norms. If the norm p is given in $\mathcal{U}$, then one can define the distance between any two elements u and $v \in \mathcal{U}$ as $p(u - v)$. We say that the sequence $(u_1, \ldots, u_n, \ldots)$ converges to the limit u if the distance between u_n and u tends to zero when $n \to \infty$, i.e., if $\lim_{n\to\infty} p(u_n - u) = 0$. Convergence defined in such a manner is sometimes called the convergence over the norm or the strong convergence.

Linear space $\mathcal{U}$ in which the convergence is given by the norm $p(u)$ is called the normalized space. Space $\mathcal{U}$ is called countably normalized (with the norms $\{p_\sigma(u)\}$, $p_1(u) \le p_2(u) \le \ldots \le p_\sigma(u) \le \ldots$) if the convergence is defined as $\lim_{\nu\to\infty} p_\sigma(u_\nu - u) = 0$ for any σ, and $u \in \mathcal{U}$. For example, the Hilbert space $L_2[a, b]$ of all complex functions $f(x)$ with norm

$$(f, f) = \|f\|^2 = \int_a^b dx\, f^*(x) f(x)$$

is linear normalized space. Other linear normalized space is the space $\mathcal{C}([a, b])$ of continuous functions on the interval $[a, b]$ with the norm

$$p(u) = \mathrm{Sup}_{x\in[a,b]}|u(x)|.$$

We consider the space $\mathcal{C}(\sigma, \rho, n)$ of complex functions of n real variables $x = x_1, \ldots, x_n$, having continuous partial derivatives up to the order σ inclusively, and decreasing faster than $|x|^{-\rho}$ together with all derivatives at infinity. In other words, for the functions $u(x)$ from $\mathcal{C}(\sigma, \rho, n)$ all the products of the type

$$x^\alpha D^\beta u(x), \quad |\alpha| \le \rho, \quad |\beta| \le \sigma \tag{99}$$

are bounded, where

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n, \quad D^\beta = \frac{\partial^{\beta_1+\cdots+\beta_n}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}. \tag{100}$$

The norm in the space $\mathcal{C}(\sigma, \rho, n)$ is given by

$$p_{\rho\sigma}(u) = \max_{\substack{|\alpha|\le\rho \\ |\beta|\le\sigma}} \mathrm{Sup}_{x\in\mathcal{R}^n} |x^\alpha D^\beta u(x)|. \tag{101}$$

The spaces of the type of $\mathcal{C}(\sigma, \rho, n)$ play an important role in the theory of distributions. In particular, the space $S = S(\mathcal{R}^n) = \mathcal{C}(\infty, \infty, \mathcal{R}^n)$ consists of all infinitely smooth functions in the n-dimensional real space $\mathcal{R}^n$, which decrease rapidly any polynomials of $(x_1^2 + x_2^2 + \cdots + x_n^2)^{-1/2}$ together with all partial derivatives at $\|x\| \to \infty$. We define the convergence in S by the countable system norms

$$p_\sigma(u) = p_{\sigma\sigma}(u) = \max_{\substack{|\alpha|\le\sigma \\ |\beta|\le\sigma}} \mathrm{Sup}_{x\in\mathcal{R}^n} |x^\alpha D^\beta u(x)|$$

where $\sigma = 1, 2, \ldots$. In general, all functions of the type $P(x_1, \ldots, x_n) \times \exp[-x_1^2/a_1^2 - \cdots - x_n^2/a_n^2]$ including Hermit–Chebyshev functions may be used as functions of S spaces, where $P(\cdots)$ is an arbitrary polynomial. Another space $D(G)$ of test functions consists of the set of infinitely smooth functions (i.e., functions having continuous partial derivatives of all orders) in $\mathcal{R}^n$, tending to zero outside of the region G. For example, the function $u(x)$ defined by the equality

$$u_a(x) = \begin{cases} \exp[a^2/(x^2 - a^2)], & \text{for} \quad -a < x < a \ \ (a > 0); \\ 0 & \text{for} \quad |x| \ge a \end{cases}$$

belongs to the space $D(a)$.

Definition. A numerical function defined in linear space $\mathcal{U}$ is called functional. The functional $F(u)$ is called linear, if for any $u, v \in \mathcal{U}$, and for any number α and β

$$F(\alpha u + \beta v) = \alpha F(u) + \beta F(v). \tag{102}$$

The functional $F(u)$ is continuous if the convergence of sequence $\{u_n\} \in \mathcal{U}$ to $u \in \mathcal{U}$ follows the convergence of sequence $F(u_n)$ to $F(u)$. If there exists a positive number C_F such as that for any $u \in \mathcal{U}$, the inequality $|F(u)| \leq C_\mathrm{F}\, p(u)$ holds. The set of all linear functionals in normalized space $\mathcal{U}$ is linear space, we call it the space conjugate to $\mathcal{U}$ and denote it by $\mathcal{U}'$. In $\mathcal{U}'$ one can also define the norm

$$p'(F) = \ \mathrm{Sup}_{\mathrm{p}(u)\leq 1}|F(u)|$$

We retain the term "generalized function" for the functionals S', while the functionals D' are called the distributions, Schartz (1950/1951), and Gel'fand and Shilov (1964,1968) calls them generalized functions of "tempered growth."

Definition. Functions disappearing outside some finite region of space are called finite functions. Closure of point sets, on which a continuous function $u(x) \neq 0$, is called the support of this function. Some properties of generalized functions are

1. Transformation of Arguments and Differentiation

   Mutually synonymous transformation $y = \varphi(x)$ $[y_1 = \varphi_1(x), \ldots]$ or $x = \varphi^{-1}(y)$ of the space $\mathcal{R}^n$ onto itself leads to

$$(f(\varphi^{-1}(y), u(y)) = \int_{\mathcal{R}^n} d^n x \ f^*(x) u(\varphi(x)) |J(\varphi)|$$

   and

$$\left(\frac{\partial f}{\partial x_k}, u\right) \equiv -\left(f, \frac{\partial u}{\partial x_k}\right) \tag{103}$$

   for any distribution f. Here $J(\varphi)$ is the Jacobian of the transformation. We consider some examples for the generalized functions of a single variable that may be defined as the derivative of usual integrable functions.

1a. The derivative of the well-known discontinuous function $\theta(x)$ equals $\delta(x)$:

$$\left(\frac{d\theta}{dx}, u\right) = -\int_{-\infty}^{\infty} dx\, \theta(x) \frac{du}{dx} = -\int_0^{\infty} dx \left(\frac{du}{dx}\right)$$
$$= u(0) = (\delta, u). \tag{104}$$

1b. The functional $(d/dx) \ln|x|$ coincides with the principal value of $1/x$ in Cauchy's sense:

$$\left(\frac{d \ln|x|}{dx}, u(x)\right) = \mathcal{P} \int_{-\infty}^{\infty} dx \frac{u(x)}{x}. \tag{105}$$

1c. We define the function $1/x^2$ as a derivative of the generalized function $-1/x$:

$$\left(\frac{1}{x^2}, u(x)\right) = \left(\frac{1}{x}, u'(x)\right) = \int_0^\infty dx \frac{u'(x) - u'(-x)}{x}. \quad (106)$$

In general,

$$x^{-n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^n}{dx^n} \ln|x|, \quad n = 1, 2, \ldots \quad (107)$$

In accordance with (103), (107), and the property

$$\ln(x + i0) = \lim_{y \to +0} \ln(x + iy) = \ln|x| + i \lim_{y \to +0} \arg(x + iy)$$
$$= \ln|x| + i\pi\theta(-x),$$

we have

$$\frac{1}{x + i0} \equiv \frac{d}{dx} \ln(x + i0) = \frac{1}{x} - i\pi\delta(x), \quad (108)$$

or in the general case

$$(x \pm i0)^{-n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^n}{dx^n} \ln(x \pm i0) = x^{-n} \pm i\pi \frac{(-1)^n}{(n-1)!} \delta^{n-1}(x). \quad (109)$$

Using (103) and (109) one gets

$$(\delta^n, u) = (-1)^n u^{(n)}(0), \quad \delta^n = \frac{d^n}{dx^n} \delta(x).$$

2. Fourier Transforms of Generalized Functions

Consider the space $\mathcal{R}^n$ with the metric

$$s^2 = g_{ij} x_i y_j = x_1 y_1 + \cdots + x_l y_l - x_{l+1} y_{l+1} - \cdots - x_n y_n \quad (110)$$

and study the Fourier transform with respect to this bilinear form.

Definition. The Fourier transform of the generalized function $f(x)$ is defined as the linear functional $\tilde{f}(p)$ on the space of the Fourier images of the test functions $u(x)$ by the formula

$$(\tilde{f}(p), \tilde{u}(p)) = (f(x), u(x)), \quad (111)$$

where $\tilde{u}(p)$ is given by

$$\tilde{u}(p) = Fu(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathcal{R}^n} d^n x \, e^{ipx} u(x), \quad (112)$$

which is also the test function $\tilde{u}(p) \in S$. The inverse transformation of (112) possesses the same properties as F, and therefore the Fourier transform realizes an isomorphism S on S.

Theorem 1. *The Fourier transform of the generalized functions defined by the formula (111) is the isomorphism of space S' onto itself. We recall that the isomorphism of the linear topological space $\mathcal{U}_1$ onto the same space $\mathcal{U}_2$ is called the mutually synonymous and mutually continuous mapping of $\mathcal{U}_1$ onto $\mathcal{U}_2$, conserving linear operations.*

We give here Fourier transformations of some generalized functions:

2a.

$$F\theta(t) = (2\pi)^{-\frac{1}{2}} \int dt\, \theta(t) e^{i\omega t} = i(2\pi)^{-\frac{1}{2}} \frac{1}{\omega + i0}$$

2b.

$$F\delta^{(n)}(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathcal{R}^n} d^n x\, \delta^{(n)}(x) e^{ipx} = (2\pi)^{-\frac{n}{2}}$$

2c.

$$\tilde{F}\left(\frac{e^{ipx}}{(2\pi)^{\frac{n}{2}}}\right) = (2\pi)^{-n} \int_{\mathcal{R}^n} d^n p\, e^{ip(a-x)} = \delta^{(n)}(x-a)$$

where $\delta^{(n)} = \delta(x_1)\cdots\delta(x_n)$. □

3. Multiplication of the Generalized Function by a Smooth Function and Their Convolution

The product of two generalized functions represents a difficult problem in the theory of the generalized functions since it is a nonlinear operation. For instance,

$$\frac{1}{x}(x\delta(x)) = \frac{1}{x}0 = 0 \neq \left(\frac{1}{x}\cdot x\right)\delta(x) = \delta(x).$$

Nevertheless, there exists a wide class of functions for which one can define their products with the generalized functions from S' in a natural manner. It is said that the function $\varphi(x)$ is a multiplier in the space S of test functions if from $u(x) \in S$ it follows that $\varphi(x)u(x) \in S$.

Definition. If $\varphi(x)$ is the multiplier, the product of $\varphi(x)$ by a generalized function $f \in S'$ is given by

$$(\varphi(x) f, u(x)) = (f, \varphi^*(x)u(x)) \tag{113}$$

for any $u(x) \in S$. By defintion, the product is commutative, i.e., $\varphi f = f\varphi$.

We now consider the operation of convolution that is widely used in the theory of the generalized functions.

Definition. The convolution for two functions $f(p)$ and $g(p)$ is given by

$$f(p) * g(p) = \int_{\mathcal{R}^n} d^n q \; f(p-q) g(q) = \int_{\mathcal{R}^n} d^n q \; f(q) g(p-q). \tag{114}$$

Using the second equality in (114) the convolution of the generalized function f with the test function $u(p) \in S$ is defined as

$$f * u(p) = (f(q), u(p-q)) = \int_{\mathcal{R}^n} d^n q \; f(q) u(p-q). \tag{115}$$

4. Division and Support of a Generalized Function

The division is inverse in operation to multiplication and leads to the study of the equation

$$\varphi(x) f = g, \tag{116}$$

where $g \in S'$ and $\varphi(x)$ are the given multiplier functions, and f is an unknown generalized function. When $\varphi(x) \neq 0$ for any x, Eq. (116) is solved elementarily. If the function $\varphi(x)$ has zeros, the problem of division becomes complicated. Consider here only the case of a single independent variable x and assume that $\varphi(x)$ in (116) has a discrete set of zeros of finite order. Thus, $xf = g$, solution of which has the form $(f, u) = Cu(0) + (g, u_1(x))$, where $u(x)$ and $u_1(x) \in S(\mathcal{R}^1)$. While the general solution of the homogeneous equation $x f_0 = 0$ is $(f_0, u) = Cu(0)$, i.e., $f_0 = C\delta(x)$. Further, by induction it is shown that the more general equation $x^m f = g$ always has a solution with respect to $f \in S'$, where $g \in S'$ and m is a natural number. An arbitrary factor of this solution is produced by a general solution of the homogeneous equation $x^m f_0 = 0$, that is,

$$f_0 = \sum_{\nu=0}^{m-1} \frac{C_\nu}{\nu!} \delta^\nu(x), \qquad \delta^\nu = \frac{d^\nu}{dx^\nu} \delta(x), \tag{117}$$

where C_ν are arbitrary constants. Furthermore, we discuss other questions of interest, including the local properties of the generalized functions. As opposed to the usual functions, which are given at each point of some set, the generalized functions are determined as a whole as values of the functional on the space of test functions. Generally speaking, they do not have definite values at separate points of space. It is said that the generalized functions f and g coincide in the region (or in the open set) G if for any test function $u(x)$ with the support on G the equality $(f, u) = (g, u)$ holds. Thus, by definition the set of points on which the generalized function f turns to zero is open. Complement of this set to the whole space $\mathcal{R}^n$ is called the support of the generalized function f.

Theorem 2. *Let f be the generalized function located at the origin of the coordinate system (i.e., $f(x) = 0$ for $x \neq 0$). Then $f(x)$ is expressed by a finite linear combination of δ function and its derivatives $\delta^\nu(x)$. This assertion is valid for both cases of generalized functions of many variables and of the distributions from $D'(\mathcal{R}^n)$, i.e., $f \in D'(\mathcal{R}^n)$,*

$$f(x) = \sum_{|\alpha| \leq N} C_\alpha D^\alpha \delta(x), \tag{118}$$

where N is the order of f, C_α are some constants, and D^α is given by the expression (100).

In scalar field theory, the causal Green function $\Delta_{\rm c}(x)$ and the Pauli–Jordan function $\Delta(x) = [\varphi(x), \varphi(y)]_-$ are generalized functions of the type $\delta(x)$ and $\delta(x)\theta(x)$. The generalized form of (118),

$$F(x) = \sum_{n=0}^{\infty} C_n (\Box l^2)^n \delta^{(4)}(x), \tag{119}$$

is called a nonlocal generalized function, properties of which are dependent on the sequence of C_n. Roughly speaking, in difference with (118), the generalized functions (119) are located in some domain characterized by the parameter l. The space of test functions for the generalized functions (119) consists of the entire functions $f(z_n, \ldots, z_n) \in \mathcal{Z}_\alpha$, $(\alpha \geq 1)$ of n complex variables $z_i = x_i + iy_i$, satisfying the conditions:

1. *For any $f(z_1, \ldots, z_n) \in \mathcal{Z}_\alpha$, there exist such positive numbers $C > 0$ and $A_j > 0$ $(j = i, \ldots, n)$ that*

$$|f(z_1, \ldots, z_n)| \leq \exp\left\{ \sum_{j=1}^{n} A_j |z_j|^\alpha \right\}.$$

2. *For any $y_1, \ldots, y_n$*

$$\int d^4x_1 \cdots \int d^4x_n |f(x_1 + iy_1, \ldots, x_n + iy_n)| < \infty.$$

On the space $\mathcal{Z}_\alpha$ one can define all nonlocal generalized functions of the type of (119) for which the Fourier transform $\tilde{F}(p^2)$ is an entire analytical function in the complex p^2 plane, with the order of growth $\rho < \frac{\gamma}{2} = \frac{\alpha}{2(\alpha-1)}$. The generalized functions (119) are considered in Efimov (1977) and Namsrai (1986). □

10.2. The Källen–Lehmann Representation

Spectral forms (8), (19), and (49) for the composed field $\Phi(x)$, described by the square-root field equations, are similar to the Källen–Lehmann representation for a complex scalar Heisenberg picture operator $\Phi(x)$, which may or not be an elementary particle field. In their original works (Källen, 1952; Lehmann, 1954), they considered the vacuum expectation value of a product

$$\langle 0|\Phi(x)\Phi^*(y)|0\rangle = \sum_n \langle 0|\Phi(x)|n\rangle\langle n|\Phi^*(y)|0\rangle, \tag{120}$$

where the sum runs over any complete set of states. Choosing these states as eigenstates of the momentum four-vector $\hat{p}^\mu$, translational invariance gives

$$\langle 0|\Phi(x)|n\rangle = \exp(ip_n \cdot x)\langle 0|\Phi(0)|n\rangle$$

and (121)

$$\langle n|\Phi^*(y)|0\rangle = \exp(-ip_n \cdot y)\langle n|\Phi^*(0)|0\rangle.$$

Substituting (121) into (120) we can transform the sum into a spectral function (for details, see Weinberg, 1995)

$$\sum_n \delta^4(p - p_n)|\langle 0|\Phi(0)|n\rangle|^2 = (2\pi)^{-3}\theta(p^0)\rho(-p^2), \tag{122}$$

with $\rho(-p^2) = 0$ for $p^2 > 0$. With this definition, Eq. (120) reads as

$$\langle 0|\Phi(x)\Phi^*(y)|0\rangle = (2\pi)^{-3}\int d^4p\int_0^\infty d\mu^2\, \exp[ip\cdot(x-y)] \times \theta(p^0)\rho(\mu^2)\delta(p^2+\mu^2) \tag{123}$$

or

$$\langle 0|\Phi(x)\Phi^*(y)|0\rangle = \int_0^\infty d\mu^2\, \rho(\mu^2)\Delta_+(x-y,\mu^2), \tag{124}$$

where $\Delta_+(x)$ is the familiar Pauli–Jordan function

$$\Delta_+(x-y,\mu^2) = (2\pi)^{-3}\int d^4p\, \theta(p^0)\delta(p^2+\mu^2)\exp[ip\cdot(x-y)]. \tag{125}$$

In just the same way, one can get

$$\langle 0|\Phi^*(y)\Phi(x)|0\rangle = \int_0^\infty d\mu^2\, \bar{\rho}(\mu^2)\Delta_+(y-x,\mu^2). \tag{126}$$

The causality requirement tells us that the commutator $[\Phi(x), \Phi^*(y)]_-$ must vanish for spacelike separation $x - y$. The vacuum expectation value of the commutator is

$$\langle 0|[\Phi(x), \Phi^*(y)]_-|0\rangle = \int_0^\infty d\mu^2(\rho(\mu^2)\Delta_+(x-y,\mu^2) - \bar{\rho}(\mu^2)\Delta_+(y-x,\mu^2)). \tag{127}$$

Thus, it is necessary that

$$\rho(\mu^2) = \bar{\rho}(\mu^2). \tag{128}$$

Using Eq. (128), the vacuum expectation of the time-ordered product (or the causal Green function) is

$$\langle 0|T\{\Phi(x)\Phi^*(y)\}|0\rangle = -i\int_0^\infty d\mu^2\,\rho(\mu^2)\Delta_c(x-y,\mu^2) \tag{129}$$

where $\Delta_c(x,\mu^2)$ is the Feynman propagator for a scalar particle of mass μ. In momentum space

$$D(p) \equiv i\int d^4x\ \exp[-ip\cdot(x-y)]\langle 0|T\{\Phi(x)\Phi^*(y)\}|0\rangle$$

and therefore

$$D(p) = \int_0^\infty d\mu^2\,\rho(\mu^2)\frac{1}{p^2+\mu^2-i\varepsilon}. \tag{130}$$

One immediate consequence of this result and the positivity of $\rho(\mu^2)$ is that $D(p)$ cannot vanish for $|p^2| \to \infty$ faster than the propagator $1/(p^2+m^2-i\varepsilon)$. The suggestion is made to include higher derivative terms in the Lagrangian, which would make the propagator vanish faster than $1/p^2$ for $|p^2| \to \infty$, but the spectral representation shows that this would necessarily entail a departure from the positivity postulates of quantum mechanics.

A sum rule for the spectral function can be obtained if we use equal-time commutation relations. If $\Phi(x)$ is a conventionally normalized canonical field operator, then

$$\left[\frac{\partial\Phi(\mathbf{x},t)}{\partial t}, \Phi^*(\mathbf{y},t)\right]_- = -i\delta^3(\mathbf{x}-\mathbf{y}). \tag{131}$$

We know that

$$\frac{\partial}{\partial x^0}\Delta_+(x-y)\,|_{x^0=y^0} = -i\delta^3(\mathbf{x}-\mathbf{y}). \tag{132}$$

So the spectral representation (127) and the commutation relations (131) together tell us that

$$\int_0^\infty \rho(\mu^2)\,d\mu^2 = 1. \tag{133}$$

This implies that for $|p^2| \to \infty$, the momentum space propagator (130) of the unrenormalized fields has the free-field asymptotic behavior

$$D(p) \to \frac{1}{p^2}.$$

This assertion is valid for the fields described by the square-root differential operator considered above.

11. VACUUM FLUCTUATIONS

In this section, corrections to the Newton and Coulomb potentials due to vacuum fluctuations are discussed. We show that very weak Yukawa-type potentials are present in addition to the $1/r$ potentials whose strengths yield repulsion with ranges inversely proportional to some mass of particles that cause the vacuum fluctuations.

Recently, several theoretical proposals have been put forward in which a treatment of the hierarchy problem in the gravity sector was changed from the extremely small Planck scale to a TeV scale (Antoniadis *et al.*, 1998a,b). Conjectures of the cosmology and phenomenology of this model are already known (Kim, 2000). Randall and Sundrum (1999a,b) proposed different models in which the background metric is curved along the extra dimension due to the negative bulk cosmological constant. One of the consequences of these approaches is that the Newton potential (law) is changed as a function of the numbers of extra dimensions. In general, deviations for the $1/r$ Newton potential, for a unit test mass are parameterized by two parameters (Kehagias and Sfetsos, 2000; Long *et al.*, 1999)

$$V(r) = -\frac{GM}{r}\left(1 + \alpha e^{-\frac{r}{\lambda}}\right), \tag{134}$$

where G is the usual Newton gravitational constant. Such potential corrections are short-ranged, thus $r \gg \lambda$ with values for λ conjectured as large as $\sim$1 mm. The value of α depends strongly on the particular model. With a supersymmetry-breaking mechanism an α of $\sim$0.6 is found (Antoniadis *et al.*, 1998a,b), whereas a dilaton model predicts an α of $\sim$45 (Antoniadis *et al.*, 1998a,b, Taylor and Veneziano, 1998). Other authors (Kehagias and Sfetsos, 2000) studied alternative models of compactification mechanism for internal dimensions in the lowest Kaluza–Klein state, with α equal to its degeneracy. This gives for an n-dimensional torus $\alpha = 4, 6, \cdots, 14$ when $n = 2, 3, \ldots, 7$.

In this section, we show that vacuum fluctuation effects can lead to similar changes of Newton's gravitational potential on equal footing corrections for the Coulomb potential. Vacuum fluctuations are a result of the quantum-mechanical mechanism of interacting massive particles with the vacuum. They imply that the particle at times gains or loses part of its total energy

$$E_{\rm p} \to E_{\rm p} \pm E_{\rm f} \quad \text{for the propagator} \quad \frac{1}{m^2 + p^2} \to \frac{1}{m^2 + p^2 \pm \mu^2}. \tag{135}$$

To maintain energy the fluctuations cancel in time, thus

$$E_{\rm p} + E_{\rm f} - E_{\rm f} = E_{\rm p} \quad \text{or} \quad \frac{1}{m^2 + p^2 + \mu^2 - \mu^2} = \frac{1}{m^2 + p^2}. \tag{136}$$

However, a gain or loss of energy is possibly fractionated, like

$$E_{\rm p} = \sqrt{E_{\rm p}} \cdot \sqrt{E_{\rm p}} = \sqrt{E_{\rm p} + E_{\rm f}^1 - E_{\rm f}^1} \cdot \sqrt{E_{\rm p} + E_{\rm f}^2 - E_{\rm f}^2} \tag{137}$$

or

$$\begin{aligned} \frac{1}{m^2 + p^2} &= \frac{1}{\sqrt{m^2 + p^2}} \frac{1}{\sqrt{m^2 + p^2}} \\ &= \frac{1}{\sqrt{m^2 + p^2 + \mu_1^2 - \mu_1^2}} \frac{1}{\sqrt{m^2 + p^2 + \mu_2^2 - \mu_2^2}}. \end{aligned} \tag{138}$$

$E_{\rm f}^i(\mu_i)$ are portions of vacuum energy expressed in terms of masses. Next, we show that these zero–zero effects yield changes of the Newton and Coulomb laws at short distances. In the language of the graviton (photon) propagator, the Newton (and the Coulomb) potential is given by, in static limit,

$$V_{\rm N}(r) = -\frac{M G_{\rm N}}{(2\pi)^3} \int d^3p\, e^{-i\mathbf{pr}} \frac{1}{\mathbf{p}^2} = -\frac{G_{\rm N} M}{4\pi r} \tag{139}$$

and

$$V_{\rm C}(r) = \pm \frac{e}{(2\pi)^3} \int d^3p\, e^{-i\mathbf{pr}} \frac{1}{\mathbf{p}^2} = \pm \frac{e}{4\pi r}, \tag{140}$$

where $G_{\rm N}/4\pi = G$. For definiteness, we consider the Newton potential case. Omitting the tensor structure the graviton propagator can be decomposed

$$\frac{1}{p^2 - i\varepsilon} = \left[\frac{1}{m^2 + p^2 - i\varepsilon}\right] + \left[\frac{1}{p^2 - i\varepsilon} - \frac{1}{m^2 + p^2 - i\varepsilon}\right] \tag{141}$$

Gravitational effects in TeV's energy region are correspondingly a sum of two terms, a *high-energy* and a *low-energy* term. The high-energy term is calculated by using the first term in the graviton propagator (141) in Feynman diagrams for gravitational processes. The low-energy term is calculated using the second term in Eq. (141). This procedure of separating the propagator into two terms (141) is in analogy with QED, where bound states are studied in external fields using relativistic calculations (Weinberg, 1995). One advantage of this procedure is that we shall also be able to calculate deviations from the $1/r$ Newton potentional due to quantum-gravitational vacuum fluctuations in very–high-energy domain. In terms of (141) the Newton potential can be rewritten in the form of a sum of two terms

$$V_{\rm N}(r) = -\frac{G_{\rm N}\cdot M}{(2\pi)^3}\int d^3p\, e^{-i\mathbf{pr}}\left\{\left[\frac{1}{\mathbf{p}^2}-\frac{1}{\mathbf{p}^2+m^2}\right]+\left[\frac{1}{\mathbf{p}^2+m^2}\right]\right\}. \qquad (142)$$

To take into account gravitational vacuum fluctuational effects due to the presence of massive elementary particles in high energy processes we can write the high energy term in Eq. (141) in the following formal form

$$\frac{1}{p^2+m^2-i\varepsilon} = \frac{1}{m^2+p^2-\mu^2+\mu^2-i\varepsilon} = \frac{1}{p^2+\mu^2-i\varepsilon}\left[\frac{1}{1+\frac{\delta}{\mu^2+p^2-i\varepsilon}}\right], \qquad (143)$$

where $\delta = m^2-\mu^2$ and we have used the identity (136). Furthermore, we assume that the mass shift in (143) is small and obtain the series in the parameter δ. Deviation from the Newton potential is given by Eqs. (142), (143), and (136)

$$V_{\rm N}^{(1)}(r) = -\frac{MG_{\rm N}}{4\pi r}\left\{1-e^{-mr}+e^{-\mu r}\left[1-\frac{\delta}{2}\frac{r}{\mu}+\frac{\delta^2}{8}\frac{r}{\mu^3}(1+\mu r)\right.\right.$$
$$\left.\left.-\frac{1}{48}\delta^3\frac{r}{\mu^5}(3+3\mu r+\mu^2r^2)+\cdots\right]\right\}. \qquad (144)$$

Assuming $\mu = m-\varepsilon$, where ε is a small parameter of the theory, one can calculate the series in the brackets of Eq. (144) up to any desired order of ε^2. It turns out that all terms proportional to any order of the parameter δ are mutually cancelled. We verify this identity up to $O(\varepsilon^5)$ orders. It means that the momentary aspect of gain and loss of energy from vacuum by a particle given by the expression (136) does not work.

Let us now consider two particle correlation-like potentials defined by the formula (138) in graviton-induced propagation processes. In this case, one can write Eq. (143) in the form

$$\frac{1}{p^2+m^2-i\varepsilon} = \left[\frac{1}{p^2+m^2-i\varepsilon}\right]^{\frac{1}{2}}\left[\frac{1}{p^2+m^2-i\varepsilon}\right]^{\frac{1}{2}}$$
$$= \frac{1}{\sqrt{\mu_1^2+p^2-i\varepsilon}}\frac{1}{\sqrt{\mu_2^2+p^2-i\varepsilon}}$$
$$\times\left[1-\frac{1}{2}\frac{\delta_1}{\mu_1^2+p^2-i\varepsilon}+\frac{3}{8}\frac{\delta_1^2}{[\mu_1^2+p^2-i\varepsilon]^2}-\cdots\right]$$
$$\times\left[1-\frac{1}{2}\frac{\delta_2}{\mu_2^2+p^2-i\varepsilon}+\frac{3}{8}\frac{\delta_2^2}{[\mu_2^2+p^2-i\varepsilon]^2}-\cdots\right] \qquad (145)$$

where $\delta_1 = m^2 - \mu_1^2$ and $\delta_2 = m^2 - \mu_2^2$. Thus, the expression (144) takes the form

$$V_{\rm N}^{(2)}(r) = -\frac{MG_{\rm N}}{4\pi r}\left\{1 - e^{-mr} + \frac{1}{\pi}\int_0^1 d\alpha\,\alpha^{-1/2}(1-\alpha)^{-1/2}e^{-\rho r}\right.$$
$$\times\left[1 - \frac{1}{2}\frac{r}{\rho}(\alpha\delta_1 + (1-\alpha)\delta_2) + \frac{1}{8}\frac{r}{\rho^3}(1+\rho r)\big[\alpha^2\delta_1^2 + (1-\alpha)^2\delta_2^2\right.$$
$$+ 2\alpha(1-\alpha)\delta_1\delta_2\big] - \frac{1}{16}\frac{r}{\rho^5}(3 + 3\rho r + r^2\rho^2)\Big[\delta_1\delta_2^2\alpha(1-\alpha)^2$$
$$\left.\left.+ \delta_1^2\delta_2\alpha^2(1-\alpha) + \frac{1}{3}\big(\delta_1^3\alpha^3 + \delta_2^3(1-\alpha)^3\big)\Big]\right]\right\} \tag{146}$$

where $\rho = \sqrt{\mu_2^2 + \alpha(\mu_1^2 - \mu_2^2)}$.
After some calculations, assuming $\varepsilon = \mu_1^2 - \mu_2^2$ is small, one gets

$$V_{\rm N}^{(2)}(r) = -\frac{MG}{r}\left\{1 - \frac{7}{2^{17}}\frac{\varepsilon^4}{m^8}e^{-mr}[m^4r^4 + 6m^3r^3 + 15m^2r^2 + 15mr]\right\}, \tag{147}$$

where $m = \sqrt{(\mu_1^2 + \mu_2^2)/2}$.

We see that multistage or correlated aspect of gain and loss of energy from vacuum to vacuum by the particle gives rise to the change in the Newton potential, where the sign of correction due to vacuum fluctuation is negative with respect to extra-dimensional contributions in (134). It should be noted that unobservable parameters μ_i^2 in Eq. (147) must be excluded from our consideration by using integration or summation, depending on whether continuum or discrete spectrum of energy was gained from vacuum by the particle. Thus, for the continuum spectra of energy, the averaged function ε^4 in (147) per mass m takes the form

$$\begin{aligned}\mathcal{F}(\varepsilon) = \langle\varepsilon^4\rangle_{\mu_2} &= \frac{2^4}{m}\int_0^m d\mu_2\left(1 - \frac{\mu_2^2}{m^2}\right)^4 \cdot m^8 \\ &= 2^4 m^8 \cdot \int_0^1 dx(1-x^2)^4 = m^8 \cdot \frac{2^{11}}{9\cdot 7\cdot 5},\end{aligned} \tag{148}$$

where we have assumed that mass values μ_i for one-stage-gain energy must belong to the interval $0 \le \mu_i \le m\,(i = 1, 2)$ and by definition $\mu_1^2 - \mu_2^2 = 2(m^2 - \mu_2^2)$. Experimental measurable quantity for the Newton potential (147) acquires the form

$$\big\langle V_{\rm N}^{(2)}(r)\big\rangle = -\frac{MG}{r}\left\{1 - \frac{1}{45}\cdot\frac{1}{2^6}\cdot e^{-mr}[m^4r^4 + 6m^3r^3 + 15m^2r^2 + 15mr]\right\}. \tag{149}$$

We now discuss another possibility for the taking away (or extract) of zero–zero effect from vacuum: the square-root form of gain or loss of vacuum energy in (137), because this effect leads to nonlinear processes. Indeed, from previous sections we know that the square-root Klein–Gordon equation (7) and its Green function equation (5) have the solutions (15) and (8) [or (13)] with the distribution (9) obeying properties (10). For the square-root propagator (8) expression (142) reads

$$V_{\mathrm{N}}(r) = -\frac{G_{\mathrm{N}}M}{(2\pi)^3}\int d^3p\, e^{-i\mathbf{pr}}\left[\left(\frac{1}{\mathbf{p}^2} - \frac{1}{m\sqrt{m^2+\mathbf{p}^2}}\right) + \left(\frac{1}{m\sqrt{m^2+\mathbf{p}^2}}\right)\right]. \tag{150}$$

Now making use of the formula (137) and decomposing square-root propagator by a rule like (143) and (145) one gets

$$V_{\mathrm{N}}^{(3)}(r) = -\frac{MG_{\mathrm{N}}}{4\pi r}\left\{1 - \frac{2}{\pi}K_1(mr) + \frac{\mu}{\pi}\frac{2}{m}\left[K_1(\mu r) - \frac{\delta}{2}\left(\frac{r}{\mu}\right)K_0(\mu r)\right.\right.$$
$$\left.\left. + \frac{1}{8}\delta^2\left(\frac{r}{\mu}\right)^2 K_1(\mu r) - \frac{1}{48}\delta^3\left(\frac{r}{\mu}\right)^3 K_2(\mu r) + \cdots\right]\right\}, \tag{151}$$

where ($\delta = m^2 - \mu^2$) and

$$V_{\mathrm{N}}^{(4)}(r) = -\frac{MG_{\mathrm{N}}}{4r\pi}\left\{1 - \frac{2}{\pi}K_1(mr) + \frac{2}{m}\frac{\sqrt{\pi}}{\pi\Gamma^2(1/4)}\int_0^1 d\alpha\,\alpha^{-\frac{3}{4}}(1-\alpha)^{-\frac{3}{4}}\cdot\rho\right.$$
$$\times\left[K_1(\rho r) - \frac{1}{2}\frac{r}{\rho}K_0(\rho r)[\alpha\delta_1 + (1-\alpha)\delta_2] + \frac{1}{8}\left(\frac{r}{\rho}\right)^2 K_1(\rho r)\right.$$
$$\times\left(\alpha^2\delta_1^2 + (1-\alpha)^2\delta_2^2 + 2\alpha(1-\alpha)\delta_1\delta_2\right) - \frac{1}{16}\left(\frac{r}{\rho}\right)^3 K_2(\rho r)$$
$$\left.\left.\times\left[\delta_1\delta_2^2\alpha(1-\alpha)^2 + \delta_2\delta_1^2\alpha^2(1-\alpha) + \frac{1}{3}\left(\alpha^3\delta_1^3 + \delta_2^3(1-\alpha)^3\right) + \cdots\right]\right]\right\} \tag{152}$$

instead of (144) and (146), respectively. Here $\delta_1 = m^2 - \mu_1^2$ and $\delta^2 = m^2 - \mu_2^2$. After some elementary calculations we have

$$V_{\mathrm{N}}^{(3)}(r) = -\frac{GN}{r}\left\{1 - \sqrt{\frac{2}{\pi mr}}e^{-mr}\left[\frac{\varepsilon_1}{2m} + \frac{\varepsilon_1^2}{2m}\left(r + \frac{1}{4m}\right)\left(1 + \frac{\varepsilon_1}{2m}\right)\right]\right\} \tag{153}$$

and

$$V_{\mathrm{N}}^{(4)}(r) = -\frac{GN}{r}\left\{1 - \sqrt{\frac{2}{\pi mr}}e^{-mr}\left(\frac{\varepsilon_2^2}{48m^3}\left(r + \frac{3}{4m}\right)\right)\right\}, \tag{154}$$

where $m - \mu = \varepsilon_1$ and $\varepsilon_2 = \mu_1^2 - \mu_2^2$ as before. Making use of the following averaged functions

$$\mathcal{F}_1(\varepsilon_1) = \langle \varepsilon_1 \rangle_\mu = \frac{1}{m}\int_0^m d\mu(m-\mu) = \frac{1}{2}m,$$
$$\mathcal{F}_2(\varepsilon_1) = \left\langle \varepsilon_1^2 \right\rangle_\mu = \frac{1}{m}\int_0^m d\mu(m-\mu)^2 = \frac{1}{3}m^2,$$
$$\mathcal{F}_3(\varepsilon_1) = \left\langle \varepsilon_1^3 \right\rangle_\mu = \frac{1}{m}\int_0^m d\mu(m-\mu)^3 = \frac{1}{4}m^3,$$

and

$$\mathcal{F}_4(\varepsilon) = \frac{2^2}{m}\int_0^m d\mu_2\left(1-\frac{\mu_2^2}{m^2}\right)^2\cdot m^4 = \frac{2^5}{5\cdot 3}m^4,$$

one gets

$$\left\langle V_{\rm N}^{(3)}(r)\right\rangle = -\frac{GN}{r}\left\{1-\sqrt{\frac{2}{\pi m r}}e^{-mr}\cdot\frac{1}{48}\left(\frac{59}{4}+11mr\right)\right\} \qquad (155)$$

and

$$\left\langle V_{\rm N}^{(4)}(r)\right\rangle = -\frac{GN}{r}\left\{1-\sqrt{\frac{2}{\pi m r}}e^{-mr}\cdot\frac{1}{90}(4mr+3)\right\}. \qquad (156)$$

We now consider other ways to take into account gravitational vacuum fluctuational effects due to self-interaction of massive elementary particles in high-energy processes, induced by the graviton propagation. Without loss of generality we choose the interaction Lagrangian

$$\mathcal{L}_{\rm in}(x) = \frac{\lambda}{3!} : \varphi^3(x) : \qquad (157)$$

Then one-loop vacuum fluctuation is given by

$$\begin{aligned} F &= \frac{(-i)^2}{2!}\langle 0|T\left\{\iint d^4x\, d^4y : \mathcal{L}_{\rm in}(x) :: \mathcal{L}_{\rm in}(y) :\right\}|0\rangle \\ &= -\lambda^2\int d^4y\, d^4x\, D_{\rm c}^3(x-y) = -V\cdot\lambda^2\int d^4x\, D_{\rm c}^3(x), \end{aligned} \qquad (158)$$

where $D_{\rm c}(x)$ is the causal Green function of a scalar particle, V is the volume of four-dimensional space. At infinite volume limit it takes the form

$$F = -(2\pi)^4\lambda^2\delta^{(4)}(0)\tilde{D}_{\rm c}^3(0). \qquad (159)$$

Here $\tilde{D}_{\rm c}(p)$ is the Fourier transform of $D_{\rm c}(x)$. We would like to study vacuum fluctuation induced by the self-interaction in the finite volume limit. By definition vacuum fluctuation, in this case, is given by

$$F_{\rm f} = -\lambda^2 \int d^4x\, D_{\rm c}^3(x) = \lambda^2 \lim_{V\to\infty} \int d^4p\, G_V(p)\tilde{D}_{\rm c}^3(p), \tag{160}$$

where $G_V(p)$ is a smeared-out delta function

$$\lim_{V\to\infty} G_V(p) = \delta^{(4)}(p).$$

Let us consider the simple regularization function

$$G_V(p) = \frac{2N^2}{\pi^2}\frac{1}{[p^2+N^2]^3}, \tag{161}$$

where $N^2 = 1/\sqrt{V}$. This smooth function obeys the Dirac δ-function properties

$$\lim_{V\to\infty} \int d^4p\, G_V(p) = 1$$

and

$$\lim_{V\to\infty} \int d^4p\, G_V(p) f(p) = f(0)$$

for some regular function $f(p)$ at the origin. To calculate the deviation of Newton potential by the formula (142) due to vacuum fluctuation induced by self-interaction of particles, one can rewrite the second term in (142) in the form

$$\frac{1}{m^2+p^2-i\varepsilon} \Rightarrow \frac{1}{m^2+p^2-i\varepsilon} - \frac{\Lambda\lambda^2}{[p^2+N^2-i\varepsilon]^3}\frac{1}{[p^2+m^2-i\varepsilon]^3}, \tag{162}$$

where $\Lambda = 2m^6N^2/\pi^2$, by using expressions (160) and (161). Contribution to the Newton potential due to the last term in (162) is given by

$$\begin{aligned} V_{\rm f}(r) &= \frac{M\cdot G_{\rm N}}{(2\pi)^3}\int d^3p\, e^{i\mathbf{pr}}\left[\frac{\Lambda\lambda^2}{[\mathbf{p}^2+N^2]^3}\cdot\frac{1}{[\mathbf{p}^2+m^2]^3}\right] \\ &= \frac{MG_{\rm N}}{4\pi r}\left\{\frac{\Lambda\lambda^2}{2\sqrt{2}N^9} r\int_{\sqrt{\frac{1}{2}}}^{\sqrt{\frac{3}{2}}} d\beta\cdot\beta^{-8}\left(\beta^2-\frac{1}{2}\right)^2\left(\frac{3}{2}-\beta^2\right)^2 e^{-\frac{Nr\beta}{\sqrt{2}}}\right. \\ &\quad \left.\times\left[105+\frac{105N}{\sqrt{2}}r\beta+\frac{45}{2}N^2r^2\beta^2+\frac{10}{2\sqrt{2}}N^3r^3\beta^3+\frac{1}{4}N^4r^4\beta^4\right]\right\}. \end{aligned} \tag{163}$$

After some calculations we have

$$V_{\rm f}(r) = -\frac{M\cdot G_{\rm N}}{4\pi r}\left\{-24\,\Lambda\lambda^2\frac{r^2}{N^8}e^{-\frac{Nr}{2}}\right\}. \tag{164}$$

Thus, expression (142) reads

$$V_{\mathrm{N}}^{(5)}(r) = -\frac{MG}{r}\left\{1 - 24\,\Lambda\lambda^2 \frac{r^2}{N^8} e^{-\frac{Nr}{2}}\right\}. \qquad (165)$$

We again obtain repulsion corrections to the Newton potential.

It is worth notice that vacuum fluctuation phenomena due to quantum gravitational interactions are very complicated nonlinear processes, like neutron-induced chain reactions in atomic nuclear and ultra–high-energy cosmic-rays-induced avalanche-type processes in the earth's atmosphere. From the physical point of view the repulsion character of the potential contributions (149), (155), (156), and (165) due to vacuum fluctuations is of interest in the avoidance (removal) of naked singularities in quantum gravity where all matter do not attach at a single point because of the repulsion potential at short distances, which gives rise to a negative pressure of vacuum. It seems that repulsion additional contribution to the Newton potential due to vacuum fluctuations does not depend on a concrete choice of the form of regularized functions (161). Let us write another form of the function (161)

$$f_{\mathrm{d}}^{(2)}(p) = \frac{6N^4}{\pi^2}[p^2 + N^2]^{-4} \qquad (166)$$

satisfying δ-function properties for $N \to 0$. For this smooth function, expression (165) acquires the form

$$V_{\mathrm{N}}^{(6)}(r) = -\frac{MG}{r}\left\{1 - \frac{269}{2^9}\Lambda^{(2)}\lambda^2 \cdot \frac{r^4}{N^8} e^{-\frac{Nr}{2}}\right\}, \qquad (167)$$

where $\Lambda^{(2)} = 6m^6 \cdot N^4/\pi^2$. We obtain again regular, negative correction to the Newton potential. The aforementioned circumstance distinguishes potential contributions (149), (155), (156), (165), and (167) from the potential (134) obtained using extra-dimensional methods, where extra-dimensional contributions to the Newton potential are positive. Typical Compton length of elementary particles is the proton one $\lambda = \hbar/m_{\mathrm{p}}c \sim 10^{-14}$ cm, and therefore all obtained corrections (149), (155), and (156) may give measurable effects only in subnuclear distances, while contributions (165) and (167) are divergent as $\sim V^2$ at the limit $V \to \infty$. This fact tells us that corrections due to vacuum fluctuations are not correlated with respect to extra-dimensional contributions like (134) at least for $n \leq 5$.

12. EXTRA-DIMENSIONAL METHOD IN THE POTENTIAL THEORY

High-dimensional method entered in the nuclear and particle theories from the early stages of development, when Kaluza (1921) and Klein (1926) attempted to unify Einstein's theory of gravitation with Maxwell's electromagnetism by introducing one additional extra-fifth-dimension of space and its compactification with respect to the usual four-dimensional space–time. Recently, this approach has received much attention in connection with the developments in string theory and in the solution of the hierarchy problem in high-energy physics. In this section, we use this method to study potentials created by different extended objects like a charged ring, disk, etc. Before calculating potentials of extended charges we present some mathematical tricks. The volume of n-dimensional momentum space is given by

$$d^n p = p^{n-1} dp \, d\Omega_n, \quad d\Omega_n = \sin^{n-2}\theta_{n-1} d\theta_{n-1} \, \sin^{n-3}\theta_{n-2} d\theta_{n-2} \cdots d\theta_1, \tag{168}$$

with $0 \le \theta_i \le \pi$, except $0 \le \theta_1 \le 2\pi$. Using the integral formula

$$\int_0^\pi d\theta \, \sin^m \theta = \sqrt{\pi} \frac{\Gamma\left(\frac{1}{2}(m+1)\right)}{\Gamma\left(\frac{1}{2}(m+2)\right)},$$

one gets

$$\Omega_n = \int d\Omega_n = 2 \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \tag{169}$$

Let us consider the propagator of the scalar particle

$$D_\mu(p) = \frac{1}{[m^2 + p^2 - i\varepsilon]^{1+\mu}}.$$

In the static limit it leads to

$$D_\mu(\mathbf{p}) = \frac{1}{[m^2 + \mathbf{p}^2]^{1+\mu}}, \tag{170}$$

where $\mathbf{p}^2 = p_1^2 + \cdots + p_n^2$ and μ is some parameter of the theory; in particular, $\mu = 0$ corresponds to the usual Klein–Gordon propagator, while the square-root propagator is the case $\mu = -1/2$. Assuming $m = 0$ one can obtain photon and graviton propagators in the static limit. Generalized n-dimensional Yukawa-type potential is given by

$$\begin{aligned} U_n(r) &= \frac{1}{(2\pi)^n} \int d^n p \frac{e^{-i\mathbf{pr}}}{[m^2 + \mathbf{p}^2]^{1+\mu}} \\ &= \frac{\Omega_{n-1}}{(2\pi)^n} \int_0^\infty dp \; p^{n-1} \cdot \int_0^\pi d\theta_{n-1} \sin^{n-2}\theta_{n-1} \, e^{-ipr\cos\theta_{n-1}}. \end{aligned} \tag{171}$$

Furthermore, using the integrals (Gradshteyn and Ryzhik, 1980)

$$\int_0^\pi d\theta_{n-1} \sin^{n-2}\theta_{n-1} \, e^{-ipr \; \cos\theta_{n-1}} = \frac{2^{\frac{n-2}{2}} \Gamma\left(\frac{n-2}{2} + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{(pr)^{\frac{n-2}{2}}} \mathcal{J}_{\frac{n-2}{2}}(pr)$$

and

$$\int_0^\infty dp\; p^{\frac{n}{2}} \frac{\mathcal{J}_{\frac{n-2}{2}}(pr)}{[m^2+p^2]^{1+\mu}} = \Lambda(mr) \equiv \frac{m^{\frac{n-2}{2}-\mu} r^\mu}{2^\mu \Gamma(1+\mu)} K_{\frac{n-2}{2}-\mu}(mr), \tag{172}$$

one gets

$$U_n(r) = \frac{\Omega_{n-1}}{(2\pi)^n} \frac{2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{r^{\frac{n-2}{2}}} \Lambda(mr), \tag{173}$$

where $\Lambda(mr)$ is given by Eq. (172) and

$$\Omega_{n-1} = \frac{2\pi^{\frac{1}{2}(n-1)}}{\Gamma\left(\frac{1}{2}(n-1)\right)}.$$

Inequality

$$-1 < \frac{n-2}{2} < 2\mu + \frac{3}{2} \tag{174}$$

is necessary to converge integrals in (172) and (173). Thus, we see that the Klein–Gordon case $\mu = 0$ is sufficient for $n = 3, 4$ spatial spaces. Beginning from $n = 5$ (space–time is six-dimensional) Klein–Gordon propagator does not work.

Finally, we obtain the n-dimensional potential corresponding to the propagator (170) in the form

$$U_n(r) = \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{r^{\frac{n-2}{2}}} \frac{m^{\frac{n-2}{2}-\mu}}{2^\mu \Gamma(1+\mu)} r^\mu K_{\frac{n-2}{2}-\mu}(mr), \tag{175}$$

where $K_n(x)$ is the MacDonald function. Let us consider the following cases:

1. $n = 3$, i.e., space–time dimension is 4 and $\mu = 0$. Since

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z};$$

therefore

$$U_3^{\mathrm{Y}}(r) = \frac{1}{4\pi r_3} e^{-mr_3} \tag{176}$$

is just the Yakawa potential, where $r_3 = \sqrt{x^2+y^2+z^2} = r$.

2. $n = 4$, i.e., space–time dimension is 5 and $\mu = 0$. In this case, we have

$$U_4^{\mathrm{Y}}(r) = \frac{m}{4\pi^2 r_4} K_1(mr_4) \tag{177}$$

Here $r_4 = \sqrt{r^2 + x_4^2}$.

3. $n = 3$ and $\mu = -1/2$. This is the square-root propagator case, which gives

$$U_3^{\mathrm{SR}}(r) = \frac{m}{2\pi^2 r_3} K_1(mr_3) \tag{178}$$

4. For the cases $n = 5$ and $n = 6$, the potentials acquire the forms

$$U_5(r_5) = \frac{r_5^{\mu-\frac{3}{2}} m^{\frac{3}{2}-\mu}}{(2\pi)^{\frac{5}{2}} 2^{\mu} \Gamma(1+\mu)} K_{\frac{3}{2}-\mu}(mr_5) \tag{179}$$

and

$$U_6(r_6) = \frac{m^{2-\mu} r_6^{\mu-2}}{(2\pi)^3 2^{\mu} \Gamma(1+\mu)} K_{2-\mu}(mr_6) \tag{180}$$

respectively. Here $r_5 = \sqrt{r^2 + x_4^2 + x_5^2}$ and $r_6 = \sqrt{r_5^2 + x_6^2}$. We see that there are more singular potentials compared to the three-dimensional case.

The generalized form of the Coulomb potential is given by the limit $m \to 0$. For example,

$$U_3^{\mathrm{C}}(r) = \frac{1}{4\pi r}, \quad U_3^{\mathrm{C(SR)}}(r) = \frac{1}{2\pi^2 r^2}, \quad U_4^{\mathrm{C}}(r) = \frac{1}{4\pi^2 r_4^2}, \tag{181}$$

and so on. We observe the following rule:

$$U_3^{\mathrm{Y}}(r) = \int_{-\infty}^{\infty} d\lambda\, U_4\big(m\sqrt{r^2+\lambda^2}\big) = \frac{1}{4\pi r} e^{-mr}$$

and

$$U_3^{\mathrm{C}}(r) = \int_{-\infty}^{\infty} d\lambda\, U_4^{\mathrm{C}}(\lambda) = \frac{1}{4\pi r}. \tag{182}$$

Here we have used the integral

$$\int_0^{\infty} \frac{d\lambda}{\sqrt{r^2+\lambda^2}} K_1\big(m\sqrt{r^2+\lambda^2}\big) = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{2mr}} K_{\frac{1}{2}}(mr).$$

Analogously, two or three steps of lowering dimensions of space give more smooth potentials. For instance,

$$U_3^{\mathrm{f}} = \int d\lambda_1 \int d\lambda_2\, U_5\left(m\sqrt{r^2+\lambda_1^2+\lambda_2^2}\right) = \frac{m^{\frac{1}{2}-\mu}}{(2\pi)^{\frac{3}{2}} 2^{\mu} \Gamma(1+\mu) r^{\frac{1}{2}-\mu}} K_{\frac{1}{2}-\mu}(mr)$$

and

$$U_3^{\mathrm{F}}(r) = \int d\lambda_1 \int d\lambda_2 \int d\lambda_3\, U_6\left(m\sqrt{r^2+\lambda^2}\right)$$
$$= \frac{2\sqrt{2} m^{\frac{1}{2}-\mu} \Gamma\left(\frac{3}{2}\right)}{(2\pi)^2 \Gamma(1+\mu) 2^{\mu} r^{\frac{1}{2}-\mu}} K_{\frac{1}{2}-\mu}(mr). \tag{183}$$

We see that these two ways of lowering dimensions by means of integration over infinite domains coincide with each other.

Now let us consider the compactification of space dimension into finite regions. Here we present different cases:

1. Let x and y be located in a disk with radius R. Then after reduction (compactification) of space dimension on this disk, we obtain its electric static potential on the symmetric axis, i.e., z axis by assuming $m = 0$:

$$U_{\mathrm{d}}^{\mathrm{C}}(z) = \frac{1}{\pi R^2}\int_0^{2\pi} d\varphi \int_0^R d\lambda \cdot \lambda \frac{e}{4\pi\sqrt{z^2+\lambda^2}} = \frac{e}{2\pi R^2}\left[\sqrt{R^2+z^2} - z\right], \tag{184}$$

   where $e/\pi R^2$ is the density of charge.
2. Analogously, the potential of a charged stick with length $2R$ located along the z axis is given by

$$U_{\mathrm{s}}^{\mathrm{C}}(r_2) = \frac{1}{2R}\int_{-R}^{R} d\lambda \frac{e}{4\pi\sqrt{r_2^2+\lambda^2}} = \frac{e}{8\pi R}\ln\left\{\frac{1+\sqrt{1+\frac{r_2^2}{R^2}}}{\sqrt{1+\frac{r_2^2}{R^2}}-1}\right\}, \tag{185}$$

   where $r_2 = \sqrt{x^2+y^2}$.
3. Finally, we can write the potential generated by a charged ring with radius R, located on the $(z = 0)$-plane. The result reads

$$U_{\mathrm{r}}^{\mathrm{C}}(z) = \frac{1}{2\pi R}\int_0^{2\pi} d\varphi \frac{Re}{4\pi\sqrt{R^2+z^2}} = \frac{e}{4\pi\sqrt{R^2+z^2}}. \tag{186}$$

We see that reduction of the space dimension is useful in getting potentials of different expended charges. These formal procedures can be applied to some other type of potential theory, for example, the two-dimensional reduction of the Yakawa potential $U^{\mathrm{Y}}(r) = (g/4\pi r)\exp(-mr)$, where g is some coupling constant, given by

$$U_1^{\mathrm{Y}}(x) = \frac{1}{\pi R^2}\int_0^{2\pi} d\varphi \int_0^R d\lambda \cdot \lambda \frac{g}{4\pi\sqrt{x^2+\lambda^2}} e^{-m\sqrt{x^2+\lambda^2}} = \frac{g}{2m\pi R^2}\left[e^{-mx} - e^{-m\sqrt{x^2+R^2}}\right]. \tag{187}$$

This formal potential form is finite at the origin $x = 0$. It should be noted that confinement-type potentials are also obtained by using compactification method for extra-dimensional formalism. One example is

$$U_3^{\mathrm{C}}(r) = \frac{1}{\pi R^2} \int d\lambda_1 \int d\lambda_2\, U_5 \left(m\sqrt{r^2 + \lambda_1^2 + \lambda_2^2} \right)$$
$$= \frac{1}{(2\pi)^{\frac{5}{2}}} \frac{2^{\frac{1}{2}-2\mu}}{\Gamma(1+\mu)\left(\mu - \frac{1}{2}\right) R^2} \left[(r^2 + R^2)^{\mu-\frac{1}{2}} - (r^2)^{\mu - \frac{1}{2}} \right]. \quad (188)$$

Parameters R and μ are defined by the experimental data on QCD measurements.

13. ELECTROMAGNETIC INTERACTIONS OF EXTENDED CHARGES

In this section, we study electromagnetic interactions of the charged ring and dipole. Simple quantum electrodynamical calculation gives restriction $\ell_1 \leq 10^{-19}$ cm on the radius of the ring. Let us consider two typical extended objects: a ring charge (closed string) and a dipole (rigid string). In accordance with the previous section, potentials are given by

$$\varphi_{\mathrm{r}}^{\ell}(r) = \frac{e}{4\pi\varepsilon_0} \cdot \frac{1}{\sqrt{\ell_1^2 + r^2 \cos^2\theta}} \quad (189)$$

and

$$\varphi_{\mathrm{d}}^{\ell}(r) \sim \frac{ed\cos\theta}{4\pi\varepsilon_0\left(r^2 + \ell_2^2\right)} \quad \left(\ell_2 = \frac{d}{2}\right). \quad (190)$$

An explicit form of the latter is

$$\varphi_{\mathrm{d}}^{\ell}(r) = \frac{e}{4\pi\varepsilon_0} \cdot \frac{\sqrt{r^2 + \ell_2^2 + rd\cos\theta} - \sqrt{r^2 + \ell_2^2 - rd\cos\theta}}{\sqrt{\left(r^2 + \ell_2^2\right)^2 + r^2 d^2 \cos^2\theta}}, \quad (191)$$

where ℓ_1 and d are the radius of the ring charge and a length of the dipole consisting of two opposite electric charges. An angle θ characterizes orientation of the ring and the dipole in space. The orientation is not important in our scheme. We assume that the ring (or the dipole) is not made of plastic or some other insulator, so that the charge and the dipole can be regarded as unfixed in place, i.e., orientation θ of the plane of the ring (or the dipole) along its central axis is a random number belonging to the interval $-1/2\pi \leq \theta \leq 1/2\pi$ with a probability distribution $w(\theta)$

$$\int_{-\pi/2}^{\pi/2} d\theta\, w(\theta) = 1, \quad \int_{-\pi/2}^{\pi/2} d\theta \cdot \theta w(\theta) = 0, \quad \int_{-\pi/2}^{\pi/2} d\theta \cdot \theta^2 w(\theta) = \text{const.} \quad (192)$$

Thus, averaged potentials for a simple form $w(\theta) = \frac{1}{2}\cos\theta$ read

$$U_{\mathrm{r}}^{\ell}(r) = \left\langle \varphi_{\mathrm{r}}^{\ell}(r) \right\rangle_{\theta} = \frac{e}{2\pi^2}\frac{1}{r} \arcsin \frac{r}{\sqrt{\ell_1^2 + r^2}} \tag{193}$$

and

$$U_{\mathrm{d}}^{\ell}(r) = \left\langle \varphi_{\mathrm{d}}^{\ell}(r) \right\rangle_{\theta} = \frac{ed}{16}\frac{1}{r^2 + \ell_2^2}, \tag{194}$$

where we have assumed $\varepsilon_0 = 1$. Formulas (193) and (194) are classical roots of the theory. Poisson equations for (193) and (194) take the form

$$\Delta U_i^{\ell}(r) = -e\rho_i^{\ell}(r), \quad i = r, d, \tag{195}$$

where

$$\rho_{\mathrm{r}}^{\ell}(r) = \frac{\ell_1}{\pi^2}\left(r^2 + \ell_1^2\right)^{-2} \tag{196}$$

and

$$\rho_{\mathrm{d}}^{\ell}(r) = \frac{d}{16}\left\{ \frac{6}{\left(r^2 + \ell_2^2\right)^2} - \frac{8r^2}{\left(r^2 + \ell_2^2\right)^3} \right\} \tag{197}$$

satisfying the conditions:

$$\int d^3r \; \rho_{\mathrm{r}}^{\ell}(r) = 1 \quad \text{and} \quad \int d^3r \; \rho_{\mathrm{d}}^{\ell}(r) = 0 \tag{198}$$

as should be. Solutions of (195) have the standard integral forms

$$U_i^{\ell}(r) = \frac{e}{4\pi}\int d^3r' \frac{\rho_i^{\ell}(\mathbf{r} - \mathbf{r}')}{|\mathbf{r}'|}, \quad (i = r, d). \tag{199}$$

The formal four-dimensional Euclidean extension of formulas (195)–(199) is useful for studying the relativistic covariant theory of extended fields. In this case, Eqs. (195) lead to the same form

$$\Box_{\mathrm{E}} U_{i\mathrm{E}}^{\ell}(x_{\mathrm{E}}) = -e\rho_{i\mathrm{E}}^{\ell}(x_{\mathrm{E}}), \tag{200}$$

with

$$\rho_{\mathrm{rE}}^{\ell}(x_{\mathrm{E},} \ell_1) = \frac{3\ell_1}{4\pi^2}\left(r_{\mathrm{E}}^2 + \ell_1^2\right)^{\frac{-5}{2}} \tag{201}$$

and

$$\rho_{\mathrm{dE}}^{\ell}(x_{\mathrm{E}}) = \frac{3L}{16\pi}\left\{ \frac{4}{\left(r_{\mathrm{E}}^2 + \ell_2^2\right)^{\frac{5}{2}}} - \frac{5r_{\mathrm{E}}^2}{\left(r_{\mathrm{E}}^2 + \ell_2^2\right)^{\frac{7}{2}}} \right\}, \quad \left(L = \frac{\pi d}{2}\right) \tag{202}$$

obeying the normalization conditions

$$\int d^4 x_{\mathrm{E}} \rho^{\ell}_{\mathrm{rE}}\,(x_{\mathrm{E}}) = 1 \quad \text{and} \quad \int d^4 x_{\mathrm{E}} \rho^{\ell}_{\mathrm{dE}}\,(x_{\mathrm{E}}) = 0. \tag{203}$$

Here $x_{\mathrm{E}} = (\mathbf{x}, x_4)$, $r^2_{\mathrm{E}} = \mathbf{r}^2 + r^2_4$, $\Box_{\mathrm{E}} = \partial^2/\partial x^{\mathrm{E}}_i \cdot \partial x^{\mathrm{E}}_i$. In this case, solutions (199) acquire the form

$$U^{\ell}_{i\mathrm{E}}(x_{\mathrm{E}}) = \frac{e}{4\pi^2} \int d^4 y_{\mathrm{E}} \frac{\rho^{\ell}_{i\mathrm{E}}(x_{\mathrm{E}} - y_{\mathrm{E}})}{y^2_{\mathrm{E}}}, \quad (i = r, d). \tag{204}$$

The nonlocal photon propagator corresponding to the potential (193) in the static limit takes the form, for the Euclidean metric,

$$\tilde{D}^{\ell}_{\mathrm{rE}}\left(p^2_{\mathrm{E}}\right) = \frac{1}{e} \int d^4 y_{\mathrm{E}} \exp\left[-i p_{\mathrm{E}} y_{\mathrm{E}}\right] U^{\ell}_{\mathrm{rE}}(y_{\mathrm{E}}) = \frac{1}{p^2_{\mathrm{E}}} \exp\left[-\ell_1 \sqrt{p^2_{\mathrm{E}}}\right] \tag{205}$$

or for the pseudo-Euclidean x space

$$D^{\ell}_{\mathrm{r}}(x) = \frac{1}{(2\pi)^4} \frac{1}{i} \int d^4 p \exp(-ipx) \tilde{D}^{\ell}_{\mathrm{r}P}(p^2), \quad \left(p^2_{\mathrm{E}} = \mathbf{p}^2 + p^2_4\right) \tag{206}$$

in accordance with the general corresponding rule where same relations exist between propagators of the photons (massive scalar particles) and the Coulomb (Yukawa) potential.

Having the basic formulas (205) and (206) in the Pseudo-Euclidean metric we can easily study nonlocal quantum field theory of the ring in accordance with the Efimov method (Efimov, 1977, 1985; Namsrai, 1986). Let us consider the nonlocal interaction of the ring charge theory. The interaction Lagrangian has the form

$$\begin{aligned} L^{\ell}_{\mathrm{in}}(x) = \frac{1}{2} e \int d^4 y_1 \int d^4 y_2\, K_{\ell}(y_1) K_{\ell}(y_2)\, \bar{\psi}(x - y_1 - y_2) \\ \times \gamma_{\mu} \cdot \psi(x - y_1 - y_2)\left[A^0_{\mu}(x - y_1) + A^0_{\mu}(x - y_2)\right] \end{aligned} \tag{207}$$

or the differential form

$$L^{\ell}_{\mathrm{in}}(x) = e \cdot \bar{\psi}(x) \gamma_{\mu} \psi(x) A^{\ell}_{\mu}(x). \tag{208}$$

The latter means that only the photon carries nonlocality in the ring theory. Here $K_{\ell}(x)$ is the nonlocal distribution given by

$$K_{\ell}(x) = K_{\ell}(\Box \ell^2) \delta^{(4)}(x) \tag{209}$$

and

$$K_{\ell}(\Box \ell^2) = \exp\left[-\frac{1}{2} \ell_1 \sqrt{-\Box}\right], \tag{210}$$

and

$$A^{\ell}_{\mu}(x) = \int d^4y \cdot K_{\ell}(x-y) A^0_{\mu}(y) \tag{211}$$

is the nonlocal photon field due to the ring charge, with the propagator

$$\begin{aligned} D^{\ell}_{\mu\nu}(x-y) &= \langle 0|T\left[A^{\ell}_{\mu}(x)A^{\ell}_{\nu}(y)\right]|0\rangle \\ &= \int d^4y_1 \int d^4y_2 \cdot K_{\ell}(x-y_1)K_{\ell}(y-y_2) \\ &\quad \times \langle 0|T\left\{A^0_{\mu}(y_1)A^0_{\nu}(y_2)\right\}|0\rangle \\ &= \frac{-g_{\mu\nu}}{(2\pi)^4 i} \int d^4k \frac{[\tilde{K}_{\ell}(k^2\ell^2)]^2}{-k^2 - i\varepsilon} e^{-ik(x-y)}, \end{aligned} \tag{212}$$

where $[\tilde{K}_{\ell}(k^2\ell^2)] = \exp[-\frac{1}{2}\ell_1\sqrt{-k^2}\,]$ in accordance with (205), and $A^0_{\mu}(x)$ is the local photon field. The nonlocal distribution $K_{\ell}(x)$ in (211) coincides with the charge density (201)

$$K_{\ell}(x) = \rho^{\ell}_r\left(x, \frac{\ell_1}{2}\right) = \frac{3\ell_1}{8\pi^2}\left(\frac{\ell_1^2}{4} - x^2\right)^{\frac{-5}{2}}, \quad \left(x^2 = x_0^2 - \mathbf{x}^2\right) \tag{213}$$

in sense of the generalized function. It is obvious that function (213) satisfies δ-function properties at the limit $\ell_1 \to 0$.

From expressions (209)–(211) we can see that we arrive at the square-root differential operator of the type $\sqrt{-\Box}$. This is the typical mathematical structure of a square-root theory, the definition of which is given in the previous sections. The form (208) of the interaction Lagrangian grants the gauge invariance of the ring theory with respect to the nonlocal gauge group

$$\begin{aligned} A^0_{\mu}(x) &\Rightarrow A^0_{\mu}(x) + \partial_{\mu} f(x), \\ \psi(x) &\Rightarrow \psi(x)\exp\left[-ie\int d^4y \cdot K_{\ell}(x-y) f(y)\right], \\ \bar{\psi}(x) &\Rightarrow \bar{\psi}(x)\exp\left[ie\int d^4y \cdot K_{\ell}(x-y) f(y)\right]. \end{aligned}$$

By knowing (208) the S matrix can be written in the form of the T products

$$S = \lim_{\Lambda\to\infty,\delta\to 0} T^{\delta}_{\Lambda} \exp\left\{ie\int d^4x\, \bar{\psi}(x)\hat{A}^{\ell}(x)\psi(x)\right\}, \tag{214}$$

where the symbol T^{δ}_{Λ} means the so-called Wick T production or T^* operation used in the local theory, and the upper and lower case Λ and δ correspond to some intermediate regularization procedures defined in Efimov (1977), which make the theory finite in all the matrix elements of the S matrix. The limits $\Lambda \to \infty$ and

$\delta \to 0$ mean a removal of regularizations. To study the perturbation series for the S matrix (214) by prescription of the usual local theory, it is necessary to change the photon propagator (in the Feynman diagrams)

$$\Delta_{\mu\nu}(x-y) \to D^{\ell}_{\mu\nu}(x-y) = -g_{\mu\nu} D^{\ell}(x-y)$$

in accordance with (212) and to keep the usual local fermion propagator

$$S(x-y) = \langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle = \frac{1}{(2\pi)^4 i}\int d^4p \cdot \frac{1}{m-\hat{p}-i\varepsilon} e^{-ip(x-y)}. \quad (215)$$

Calculation of the matrix elements of the S matrix (214) will be carried out by standard method. Here we write down some corrections to the quantum electrodynamical quantities.

At present, experimental values (Caso *et al.*, 1998) of the anomalous magnetic moment of the electron and the muon are

$$\Delta\mu^{(e)}_{\rm exp} = 1.001159652193 \pm 1 \times 10^{-11}$$

and (216)

$$\Delta\mu^{(\mu)}_{\rm exp} = 1.001165923 \pm 84 \times 10^{-10}$$

and are fully described by the local QED (Kinoshita, 1988; Kinoshita *et al.*, 1984; Schwinberg *et al.*, 1981). For example, theoretical calculation for $a_{\rm e}(\text{theor}) = [(g-2)/2]_{\rm th}$ is given by

$$a_{\rm e}(\text{theor}) = a^1_{\rm e}(\text{theor}) + a^2_{\rm e}(\text{theor}). \quad (217)$$

The first term is due to pure electromagnetic local interaction, which is defined as (Schwinberg *et al.*, 1981)

$$a^1_{\rm e}(\text{theor}) = (1159652411 \pm 166)10^{-12}$$
$$= \frac{\alpha}{2\pi} - 0.328478455\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4,$$

where

$$C_3 = 1.1765 \pm 0.0013, \quad C_4 = -0.8 \pm 2.5, \quad \alpha^{-1} = 137.035963 \pm 15 \times 10^{-6}$$

so that

$$a^1_{\rm e}(\text{theor}) = 0.001159652455 \pm 127 \times 10^{-12} \pm 17 \times 10^{-12} \pm 73 \times 10^{-12}. \quad (218)$$

Term $\pm 127 \times 10^{-12}$ is due to the fine-structure constant's error, and other two errors are responsible for coefficients C_3 and C_4 respectively. The second term in (217) denotes contributions of QED breakdown effects, including the size of the electron and hadronic and Z^0–boson exchange corrections (Arzt *et al.*, 1994; DeRafael, 1994; Einhorn, 1994; Ford *et al.*, 1983; Rodrigues *et al.*, 1993). In particular, in our case

$$a_e^2(\text{theor}) \Rightarrow a_e^{\text{st}} = -\frac{\alpha}{2\pi} \cdot \frac{8}{15} m\ell_1. \tag{219}$$

The correction of the linear order of $m\ell_1$ in (219) differs essentially from the nonlocal theory (Efimov, 1977; Namsrai, 1986) in form factors and is caused by the square-root differential operator (210), which is the main characteristic in the interactions of the charged ring.

Since experimental values (216) are described by the local quantum electrodynamic expressions (218), by comparing the corrections (219) with the experimental errors in (216) one can obtain

$$\ell_1^{(e)} \leq 6.2 \times 10^{-19}\text{cm} \quad \text{for} \quad \Delta\mu_{\text{exp}}^{(e)},$$

and (220)

$$\ell_1^{\mu} \leq 2.4 \times 10^{-18}\text{cm} \quad \text{for} \quad \Delta\mu_{\text{exp}}^{(\mu)}.$$

Ring charge correction to the Lamb shift (Brodsky and Drell, 1970) is given by

$$\Delta E_\ell\left(2S_{\frac{1}{2}} - 2P_{\frac{1}{2}}\right) = \frac{11}{15} \cdot \frac{\alpha^3}{2\pi} Ry \cdot m\ell_1. \tag{221}$$

From this we conclude that

$$\ell_1^{(e)} \leq 2.9 \times 10^{-15}\text{cm}. \tag{222}$$

A ratio of cross-sections for the electromagnetic processes $e^-e^- \to e^-e^-$, $e^+e^- \to e^+e^-$, $e^+e^- \to \mu^+\mu^-$, calculated by local and nonlocal ring theories, is

$$\frac{\sigma_{\text{nonlocal}}}{\sigma_{\text{local}}} = \left[V_\ell\left(-s\ell_1^2\right)\right]^2 \approx 1 - 2\sqrt{s}\ell_1 \tag{223}$$

in accordance with formula (212) where $V_\ell(k^2\ell^2) = [K_\ell(k^2\ell^2)]^2$. Here $s = (p_1 + p_2)^2 = W^2$; W is the centre-of-mass energy. Experimental data (L3 Collaboration, 1993) gives the restriction on ℓ_1:

$$\ell_1 \leq 6.0 \times 10^{-18}\text{cm}. \tag{224}$$

All these bounds, (220), (222), and (224), mean that the radius of the charged ring for leptons is smaller than $\sim 10^{-19}$cm, so that QED is almost local theory.

14. THE SQUARE-ROOT NONLOCAL QUANTUM ELECTRODYNAMICS

In previous sections we have proposed a simple method allowing us to work with the square-root operator and to give its physical interpretation. The purpose of this section is to study local and nonlocal electromagnetic interactions of charged spinors, with photons within this scheme. Thus, the Lagrangian corresponding to Eq. (7) is given by

$$L^o_\varphi = \varphi^*(x)\sqrt{m^2 - \Box}\varphi(x). \tag{225}$$

Instead of (225) we consider the Lagrangian density

$$L^o_\psi = -N\left\{\bar{\psi}(x, \lambda_1)(-\hat{\partial})\psi(x, \lambda_2) + L^o_{1\psi}\right\} \tag{226}$$

for the $\psi(x, \lambda)$ field. Here notations

$$\begin{aligned} L^o_{1\psi} &= \bar{\Psi}(x, \lambda_1)U(\lambda_1, \lambda_2)\Psi(x, \lambda_2), \\ N &= \int_{-m}^{m}\int_{-m}^{m} d\lambda_1\, d\lambda_2\, \rho(\lambda_1)\rho(\lambda_2), \quad \hat{\partial} = i\gamma^\mu \frac{\partial}{\partial x^\mu}, \end{aligned} \tag{227}$$

$$\bar{\Psi}(x, \lambda_1) = (0, \bar{\psi}(x, \lambda_1)),$$

$$\Psi(x, \lambda_2) = \begin{pmatrix} \psi(x, \lambda_2) \\ 0, \end{pmatrix}$$

and

$$U(\lambda_1, \lambda_2) = \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix}$$

are used. Equations of motion

$$\int_{-m}^{m} d\lambda\, \rho(\lambda)(\hat{\partial} - \lambda)\psi(x, \lambda) = 0, \quad \text{and}$$

$$\int_{-m}^{m} d\lambda\, \rho(\lambda)\left(i\frac{\partial\bar{\psi}(x, \lambda)}{\partial x^\nu}\gamma^\nu + \lambda\bar{\psi}(x, y)\right) = 0 \tag{228}$$

for the $\psi(x, \lambda)$ fields can be obtained from the action

$$A = \int d^4x\, L^o_\psi(x),$$

by using independent variations over the fields $\psi(y, \lambda)$ and $\bar{\psi}(y, \lambda)$ and by taking the difference between $\delta L^o_{1\psi}/\delta\bar{\psi}(y, \lambda)$ and $\delta(L^o_{1\psi})^T/\delta\psi(y, \lambda)$. Here we have used the following obvious relations

$$\frac{\delta\bar{\psi}(x, \lambda_i)}{\delta\bar{\psi}(y, \lambda)} = \frac{\delta\psi(x, \lambda_i)}{\delta\psi(y, \lambda)} = \delta^{(4)}(x - y)\delta(\lambda_i - \lambda)$$

and definition

$$\left(L^o_{1\varphi}\right)^T = \bar{\Psi}(x, \lambda_1)U^T(\lambda_1, \lambda_2)\Psi(x, \lambda_2).$$

It is easily seen that the propagator of the field $\varphi(x)$ in (7) is given by Eq. (5) or

$$D(x) = -\frac{1}{\sqrt{m^2 - \Box}}\delta^{(4)}(x) = \frac{1}{i}\int_{-m}^{m} d\lambda\ \rho(\lambda)\,\frac{1}{\lambda + \hat{\partial}}\delta^{(4)}(x) = \int_{-m}^{m} d\lambda\ \rho(\lambda) S(x, \lambda). \tag{229}$$

In the momentum representation expression (229) takes the form [see Section 1, where $\tilde{\Omega}_c(p) \equiv \tilde{D}(p)$]

$$\tilde{D}(p) = \int_{-m}^{m} d\lambda\ \rho(\lambda)\tilde{S}(\lambda,\, \hat{p}), \tag{230}$$

where

$$\tilde{S}(\lambda,\, \hat{p}) = \frac{1}{i}\frac{\lambda + \hat{p}}{\lambda^2 - p^2 - i\varepsilon} \tag{231}$$

is the spinor propagator with mass λ in momentum space.

14.1. Square-Root Local Quantum Electrodynamics

Introducing electromagnetic interaction into the square-root formalism is same as in the local theory. To ensure invariance of the Lagrangian (226) with respect to the local gauge transformation

$$\psi'(x, \lambda) = e^{ief(x)}\psi(x, \lambda)$$

and (232)

$$\overline{\psi}'(x, \lambda) = e^{-ief(x)}\overline{\psi}(x, \lambda),$$

the gauge field $A_\mu(x)$ should be introduced into it with the transformation rule

$$A'_\mu(x) = A_\mu(x) + \frac{\partial f}{\partial x^\mu}. \tag{233}$$

As a rule, the standard procedure of changing $\partial_\mu\psi \to (\partial_\mu - ieA_\mu)\psi$ in (226) leads to the interaction Lagrangian

$$L_{\rm in}(x) = eN\{\overline{\psi}(x, \lambda_1)\hat{A}(x)\psi(x, \lambda_2)\} \tag{234}$$

in our case, where $\hat{A} = \gamma^\mu A_\mu(x)$ and N is given by (227). With (234) the S matrix can be constructed by the usual rule:

$$S = \text{Expec}_{\{\lambda_i\}}.\, T \exp\left\{\int d^4x L_{\rm in}(x)\right\}, \tag{235}$$

where the symbol T is defined by (17) for the spinor fields. Expec means to take expectation value over the random variables λ_i. In the square-root formalism, random variables λ_i entering into the definition of the spinor propagator with mass λ_i are not independent and have strong correlations between them. In other words, functions $S(x, \lambda_i)$ are some stochastic processes over the variable λ_i. Expectation values of these processes are defined by the requirement of the gauge invariance of the theory and possess some properties like white noise. For example, at least for connected diagrams in the momentum space one assumes

$$\text{Expec}\{\tilde{D}(p)\} = \int_{-m}^{m} d\lambda\, \rho(\lambda)\tilde{S}(\hat{p}, \lambda), \tag{236}$$

$$\begin{aligned}&\text{Expec}\,\{\gamma^{\nu_1}\tilde{D}(p_1)\gamma^{\nu_2}\tilde{D}(p_2)\gamma^{\nu_3}\}\\ &\quad= \frac{1}{2}\int_{-m}^{m}\int_{-m}^{m} d\lambda_1\, d\lambda_2\, \rho(\lambda_1)\rho(\lambda_2) \times \{\gamma^{\nu_1}\tilde{S}(\hat{p}_1, \lambda_1)\gamma^{\nu_2}\tilde{S}(\hat{p}_2, \lambda_2)\gamma^{\nu_3}\}\\ &\qquad\times\left\{\frac{\delta(\lambda_1-\lambda_2)}{\rho(\lambda_2)} + \frac{\delta(\lambda_1-\lambda_2)}{\rho(\lambda_1)}\right\} = \int_{-m}^{m} d\lambda\, \rho(\lambda)\{\gamma^{\nu_1}\tilde{S}(\hat{p}_1, \lambda)\gamma^{\nu_2}\tilde{S}(\hat{p}_2, \lambda)\gamma^{\nu_3}\}\end{aligned}$$

and so on. In the general case, one gets

$$\begin{aligned}&\text{Expec}\{\gamma^{\nu_1}\tilde{D}(p_1)\gamma^{\nu_2}\cdots\gamma^{\nu_n}\tilde{D}(p_n)\gamma^{\nu_{n+1}}\}\\ &\quad= \int_{-m}^{m} d\lambda\, \rho(\lambda)\{\gamma^{\nu_1}\tilde{S}(\hat{p}_1, \lambda)\gamma^{\nu_2}\cdots\gamma^{\nu_n}S(\hat{p}_n, \lambda)\gamma^{\nu_{n+1}}\}\end{aligned} \tag{237}$$

Definitions (236) and (237) grant the gauge invariance of the local quantum electrodynamics in the square-root formalism. Indeed, in the language of the perturbation theory (or the Feynman diagrammatical techniques) the gauge invariance of the "square-root" local QED means that every matrix element of the S matrix (235) defining the concrete electromagnetic processes has a definite structure, and algebraical relations exist between them. In particular, in the momentum representation, the so-called vacuum polarization diagram in the second order of the perturbation theory has the form

$$\tilde{\Pi}_{\mu\nu}(k) = (k_\mu k_\nu - g_{\mu\nu}k^2)\Pi(k^2) \tag{238}$$

and the relation

$$\frac{\partial\tilde{\Sigma}(p)}{\partial p_\mu} = -\tilde{\Gamma}_\mu(p, q)\,|_{q=0} \tag{239}$$

is valid between the vertex function $\tilde{\Gamma}_\mu(p, q)$ and the self-energy of the "sqr electron" $\tilde{\Sigma}(p)$. The relation (239) generalizes the Ward–Takahashi identity in

QED. Here in accordance with (236) and (237) we have

$$\tilde{\Sigma}(p) = \frac{-ie^2}{(2\pi)^4}\int_{-m}^{m} d\lambda\,\rho(\lambda)\int d^4k\Delta(k^2)\gamma^\mu\tilde{S}(\hat{p}-\hat{k},\lambda)\gamma^\mu \tag{240}$$

and

$$\begin{aligned}\tilde{\Gamma}_\mu(p,q) &= \frac{ie^2}{(2\pi)^4}\int d^4k\Delta((p-k)^2)\,\text{Expec}\{\gamma^\nu\tilde{D}(q+k)\gamma^\mu\tilde{D}(k)\gamma^\nu\}\\ &= \frac{ie^2}{(2\pi)^4}\int_{-m}^{m} d\lambda\,\rho(\lambda)\int d^4k\Delta((p-k)^2)\gamma^\nu\tilde{S}(\hat{q}+\hat{k},\lambda)\gamma^\mu\tilde{S}(\hat{k},\lambda)\gamma^\nu,\end{aligned} \tag{241}$$

where $\tilde{S}(\hat{p},\lambda) = (\lambda-\hat{p})^{-1}$ and $\Delta(k^2) = (-k^2 - i\varepsilon)^{-1}$. For the proof of the relation (239) consider the identity

$$\frac{\partial\tilde{S}(\hat{p},\lambda)}{\partial p_\mu} = \tilde{S}(\hat{p},\lambda)\gamma^\mu\tilde{S}(\hat{p},\lambda). \tag{242}$$

Further, it is easy to verify the identify (239) by differentiating (240) over p_μ and making use of the equality (242) as well as choosing other momentum variables in (241) and assuming $q = 0$, $p' = p + q = p$. The relations of the type $q_\mu\tilde{\Gamma}_\mu(p,q)\mid_{p'^2=p^2=\lambda^2} = 0$ follows from the definition

$$\begin{aligned}q_\mu\,\text{Expec}\{\tilde{D}(p_1)\gamma^\mu\tilde{D}(p_2)\} &= q_\mu\int_{-m}^{m}\int_{-m}^{m} d\lambda_1\,d\lambda_2\,\rho(\lambda_1)\rho(\lambda_2)\\ &\quad\times\tilde{S}(\hat{p}_1,\lambda_1)\gamma^\mu\tilde{S}(\hat{p}_2,\lambda_2)\frac{\delta(\lambda_1-\lambda_2)}{\rho(\lambda_1)}\\ \tilde{D}(p_1)-\tilde{D}(p_2) &= \int_{-m}^{m} d\lambda\,\rho(\lambda)[\tilde{S}(\hat{p}_1,\lambda)\\ &\quad-\tilde{S}(\hat{p}_2,\lambda)]\quad\text{if } q = p_1 - p_2.\end{aligned} \tag{243}$$

Now let us demonstrate that the gauge invariance of the photon self-energy diagram in the "square-root" QED and matrix element is given by

$$\begin{aligned}\tilde{\Pi}_{\mu\nu}(k) &= e^2\,\text{Expec}\left\{\int d^np\,\text{Tr}[\gamma^\mu\tilde{D}(p+k)\gamma^\nu\tilde{D}(p)]\right\}\\ &= e^2\int_{-m}^{m} d\lambda\,\rho(\lambda)\int d^np\,\text{Tr}[\gamma^\mu\tilde{S}(\hat{p}+\hat{k},\lambda)\gamma^\nu\tilde{S}(\hat{p},\lambda)].\end{aligned} \tag{244}$$

Here we have used the n-dimensional gauge-invariant regularization procedure due to 't Hooft and Veltman (1972) and the definition (237). After some calculations we obtain the same form as (238):

$$\Pi_{\mu\nu}(k) = \frac{8i\pi^{\frac{n}{2}}}{\Gamma(2)}\Gamma\left(2-\frac{n}{2}n\right)(k_\mu k_\nu - k^2 g_{\mu\nu})\int_{-m}^{m} d\lambda\,\rho(\lambda) \times \int_0^1 dx \cdot x(1-x)[\lambda^2 - k^2 x(1-x)]^{\frac{n}{2}-2}, \tag{245}$$

which is manifestly gauge-invariant. Investigation of the higher order matrix elements for the S matrix (235) can be carried out by a similar method as this.

14.2. Square-Root Nonlocal Quantum Electrodynamics

14.2.1. A Charge Distribution and Modification of the Photon Propagator

Distributional or nonlocal character of the propagator (28) and the wave function (29) calls for introducing the hypothesis that an electric charge of charged "square-root" particles (say, electrons) is not pointlike and has also some distribution $\rho(\mathbf{r})$ over space. To realize this idea, we should smear out the infinitely sharp δ function involved in the definition of the idealized concept of a pointlike charge by the following change

$$e\delta(\mathbf{r}) \Rightarrow e\rho_\ell(\mathbf{r}),$$

where a first consistent scheme is

$$\lim_{\ell\to 0}\rho_\ell(\mathbf{r}) = \delta(\mathbf{r}). \tag{246}$$

Here the distribution $\rho_\ell(\mathbf{r})$ describes the extended electric charge due to the existence of some parameter, ℓ dimension of length, which we call the fundamental length. Of course, the form of the distribution $\rho_\ell(\mathbf{r})$ is different from (9) or (30).

In a previous work (Namsrai, 1996c) we have found explicit form of this distribution

$$\rho_\ell(\mathbf{r}) = \frac{1}{\pi^{\frac{3}{2}}\ell^3}\exp(-\mathbf{r}^2/\ell^2), \tag{247}$$

which leads to the "nonlocal" Poisson equation

$$\Delta\varphi_\ell(\mathbf{r}) = -e\rho_\ell(\mathbf{r}), \tag{248}$$

and its solution is the nonlocal Coulomb potential

$$\varphi_\ell(\mathbf{r}) = \frac{e}{4\pi}\int d\mathbf{r}'\,\frac{\rho_\ell(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}'|}. \tag{249}$$

This potential is, in turn, related to a nonlocal photon propagator by

$$\Delta_\ell(\mathbf{p}^2) = \frac{1}{e}\int d^3r\, e^{-i\mathbf{pr}}\varphi_\ell(\mathbf{r}) \tag{250}$$

in the static limit. As shown in Namsrai (1996c) with the choice of the nonlocal photon field

$$A_\mu^\ell(x) = \int d^4y\, \rho_\ell^2(x-y) A_\mu(y) \tag{251}$$

the photon propagator

$$\begin{aligned}\Delta_{\mu\nu}^\ell(x-y) &= \langle 0|T\left\{A_\mu^\ell(x)A_\nu^\ell(y)\right\}|0\rangle \\ &= \frac{i}{(2\pi)^4} g_{\mu\nu} \int d^4p\, e^{ip(x-y)} \frac{\left[\tilde{\rho}_\ell^2(p)\right]^2}{-p^2 - i\varepsilon}\end{aligned} \tag{252}$$

coincides with (250) in the static limit for the momentum space. Here

$$\rho_\ell^2(x) = \frac{i}{(2\pi)^4} \int d^4p\, e^{-ipx} \tilde{\rho}_\ell^2(p) = \exp\left[\frac{\Box \ell^2}{8}\right] \delta^{(4)}(x) \tag{253}$$

is the generalized distribution, and the Fourier transform is defined as

$$\tilde{\rho}_\ell^2(p) = \exp\left[\frac{-p_E^2 \ell^2}{8}\right], \quad p_E^2 = p_4^2 + \mathbf{p}^2$$

in the Euclidean space. So that

$$\tilde{\rho}_\ell(\mathbf{p}) = \left[\tilde{\rho}_\ell^2(\mathbf{p}, p_0)\right]^2 |_{P_0=0} \tag{254}$$

as it should be. In (254) $\tilde{\rho}_\ell(\mathbf{p})$ is the Fourier transform of the charge distribution (247). The extended form of the charge distribution (253) in the Minkowski space–time is just the generalized function investigated in Efimov (1977, 1985). We see that the modified Coulomb law (249) with (247) is

$$\varphi_\ell(r) = (e/4\pi r)\phi(r/\ell), \tag{255}$$

where $\phi(x)$ is the probability integral

$$\phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{-t^2}.$$

It is natural that in our theory, the self-energy of the extended charge is finite

$$W_\ell = \frac{e}{2} \int d^3r\, \rho_\ell(r)\, \varphi_\ell(r) = \frac{\alpha}{\ell} \frac{1}{\sqrt{2\pi}}, \quad \alpha = \frac{e^2}{4\pi} \tag{256}$$

and the photon propagator $\Delta_{\mu\nu}^\ell(x) = -g_{\mu\nu}\Delta_\ell(x)$ has no singularities at the point $x = 0$,

$$\Delta_\ell(0) = \frac{1}{(2\pi)^4} 2\pi^2 \int_0^\infty du\, \frac{u}{2}\, e^{\frac{-u\ell^2}{4}} \cdot \frac{1}{u} = \frac{1}{4\pi^2} \frac{1}{\ell^2}$$

Our next aim is to construct nonlocal electromagnetic interaction of quantized fields (15) (or (29)) and (251) with the propagators (236) (or (28)) and (252), respectively. We call this scheme the square-root nonlocal quantum electrodynamics.

14.2.2. A Nonlocal Gauge Transformation and the Square-Root Nonlocal Electromagnetic Interaction

In the nonlocal case, instead of (232) and (233) we use the following nonlocal gauge transformations in accordance with the charge distribution (247) and the nonlocal photon field (251):

$$A_\mu(x) \to A_\mu(x) + \partial_\mu f(x),$$
$$\psi(x) \to \psi(x)\exp\left[ie\int d^4y\,\rho_\ell^2(x-y)f(y)\right],$$

and (257)

$$\bar{\psi}(x) \to \bar{\psi}(x)\exp\left[-ie\int d^4y\,\rho_\ell^2(x-y)f(y)\right].$$

This transformation gives rise to the change

$$\partial_\mu\psi \to \left(\partial_\mu - ie\int d^4y\,\rho_\ell^2(x-y)A_\mu(y)\right)\psi(x)$$

and (258)

$$\partial_\mu\bar{\psi} \to \left(\partial_\mu - ie\int d^4y\,\rho_\ell^2(x-y)A_\mu(y)\right)\bar{\psi}(x)$$

in the Lagrangian density (226). The change (258) in turn gives the interaction Lagrangian

$$L_{\rm in}^\ell(x) = eN\{\bar{\psi}(x,\lambda_1)\hat{A}^\ell(x)\psi(x,\lambda_2)\} \qquad (259)$$

instead of (234). Here $\hat{A}^\ell = \gamma^\mu A_\mu^\ell(x)$ and

$$A_\mu^\ell(x) = \int d^4y\,\rho_\ell^2(x-y)A_\mu(y).$$

e is the electron ("square-root") charge and $\rho_\ell(x)$ is its distribution.

Formally, the S matrix can be written in the form of T products as in (235):

$$S = \text{Expec}_{\{\lambda_i\}}\,T\exp\left\{\int d^4x\,L_{\rm in}^\ell(x)\right\} \qquad (260)$$

Investigation of matrix elements for this S matrix is similar to the local case (for details, see Namsrai, 1996c) and Section 14.1.

The construction of the perturbation series for the S matrix (260) is possible only within the framework of a regularization procedure. In the square-root nonlocal quantum electrodynamics it is sufficient to regularize the nonlocal photon propagator (252) and closed fermion loops. Thus, for the regularized photon propagator in momentum space one gets

$$\triangle^{\delta}_{\text{ph}}(k^2) = \frac{\ell^2}{2i}\int_{-\alpha+i\infty}^{-\alpha-i\infty} d\xi\, \frac{v(\xi)}{\sin\pi\xi} e^{\delta\xi^2}[\ell^2(-k^2 - i\varepsilon)]^{\xi-1}, \tag{261}$$

where we have used the Mellin representation

$$V(-p^2\ell^2) = \exp\left[\frac{p^2\ell^2}{4}\right] = \frac{1}{2i}\int_{-\alpha+i\infty}^{-\alpha-i\infty} d\xi\, \frac{v(\xi)}{\sin\pi\xi}\ell^{2\xi}[-p^2]^{\xi},$$

$$v(\xi) = \frac{1}{\Gamma(1+\xi)}2^{-2\xi}, \quad 0 < \alpha < 1 \tag{262}$$

for the Fourier transform of the charge density $[\rho^2_\ell(x)]^2$. Closed fermion loops are studied by the same method as in Section 14.1.

14.2.3. The Calculation of the Primitive Feynman Diagrams in the Square-Root Nonlocal QED

Here for pure illustrative purposes we consider simple matrix elements for the S matrix (260) corresponding to diagrams of the self-energy and vertex function for the "square-root" electron. Thus, the corresponding term in the S matrix for the diagram of self-energy is given by

$$\int_{-m}^{m} d\lambda_2 \int_{-m}^{m} d\lambda_3\, \rho(\lambda_2)\rho(\lambda_3)\{-i : \bar{\psi}(x,\lambda_2)\Sigma_\ell(x-y)\psi(y,\lambda_3) :\}, \tag{263}$$

where

$$\Sigma_\ell(x-y) = \int_{-m}^{m} d\lambda_1\, \rho(\lambda_1)\{-ie^2\gamma_\mu S(x-y,\lambda_1)\gamma_\mu\Delta_\ell(x-y)\}.$$

Passing to the momentum representation and making use of our regularization procedure δ that allows us to go to the Euclidean metric by using $k_0 \to \exp[i\pi/2]k_4$, one gets

$$\begin{aligned}\tilde{\Sigma}_\ell(p) &= \int_{-m}^{m} d\lambda\, \rho(\lambda) \lim_{\delta\to 0}(-ie^2)\int d^4x\, e^{ipx}\gamma_\mu S(x,\lambda)\gamma_\mu\Delta^{\delta}_\ell(x)\\ &= \int_{-m}^{m} d\lambda\, \rho(\lambda)\frac{e^2}{(2\pi)^4}\int d^4k_{\text{E}}\frac{V\left(k^2_{\text{E}}\ell^2\right)}{k^2_{\text{E}}}\gamma^{(\text{E})}_\mu\frac{\lambda - \hat{p}_{\text{E}} + \hat{k}_{\text{E}}}{\lambda^2 + (p_{\text{E}} - k_{\text{E}})^2}\gamma^{(\text{E})}_\mu.\end{aligned} \tag{264}$$

Here $p_{\text{E}} = (-ip_0, \mathbf{p})$, $\gamma^{(\text{E})} = (-i\gamma_0, \gamma)$, and $k_{\text{E}} = (k_4, \mathbf{k})$. Taking into account the Mellin representation (262) for the form factor $V(k^2_{\text{E}}\ell^2)$, and after some calculations, we have

$$\tilde{\Sigma}_\ell = \int_{-m}^{m} d\lambda\, \rho(\lambda) \frac{e^2}{8\pi} \frac{1}{2i} \int_{-\alpha+i\infty}^{-\alpha-i\infty} \frac{d\xi}{(\sin\pi\xi)^2} \frac{v(\xi)(\lambda^2\ell^2)^\xi}{\Gamma(1+\xi)} F(\xi, p), \tag{265}$$

where

$$F(\xi, p) = \frac{1}{\Gamma(1-\xi)} \int_0^1 du \left(\frac{1-u}{u}\right)^\xi \left(1 - \frac{p^2}{\lambda^2}u\right)^\xi (2\lambda - \hat{p}u) \tag{266}$$

is a regular function in the half-plane $Re\xi > -1$. Assuming the value $\lambda^2\ell^2$ to be small, one can obtain

$$\begin{aligned}
\tilde{\Sigma}_\ell(p) = \int_{-m}^{m} d\lambda\, \rho(\lambda) \Bigg\{ & \frac{e^2}{8\pi^2} \int_0^1 du(2\lambda - u\hat{p}) \ln\left(1 - \frac{p^2}{\lambda^2}u\right) - \frac{e^2}{16\pi^2} \\
& \times \left[\left(3\ln\frac{1}{\lambda^2\ell^2} + 3v'(0) + 3\psi(1) + 1\right) + 4\lambda^2\ell^2 v(1)\left(\ln\frac{1}{\lambda^2\ell^2} - \frac{v'(1)}{v(1)}\right.\right. \\
& \left.\left. - \frac{5}{12}\frac{p^2}{\lambda^2}\right)\right] - \frac{e^2}{16\pi^2}(\lambda - \hat{p})\left[\left(\ln\frac{1}{\lambda^2\ell^2} - v'(0) + 1\right) - \lambda^2\ell^2 v(1)\frac{p^2}{3\lambda^2}\right] \\
& + O((\lambda^2\ell^2)^2)\Bigg\}.
\end{aligned} \tag{267}$$

Let us calculate the correction to the electron mass

$$\begin{aligned}
\partial m &= \int_{-m}^{m} d\lambda\, \rho(\lambda)(\lambda_0 - \lambda) = -\int_{-m}^{m} d\lambda\, \rho(\lambda)\tilde{\Sigma}(\lambda) \\
&= \frac{3}{4\pi}\alpha \int_{-m}^{m} d\lambda\, \rho(\lambda)[\chi + 0(1)]\lambda = 0,
\end{aligned} \tag{268}$$

where $\chi = \ln[1/(\lambda^2\ell^2)]$. Expression (268) means that in the square-root QED the electron mass is not of electromagnetic origin.

Analogously, in accordance with the Expectation rule (237) in the momentum space and in the Euclidean metric, the vertex function takes the form

$$\begin{aligned}
\tilde{\Gamma}_\mu(p_1, p) = \int_{-m}^{m} d\lambda\, \rho(\lambda) \Bigg\{ & -\frac{e^2}{(2\pi)^4} \int d^4k_{\mathrm{E}} \frac{V((p_{\mathrm{E}} - k_{\mathrm{E}})^2\ell^2)}{(p_{\mathrm{E}} - k_{\mathrm{E}})^2} \\
& \times \gamma_\nu \frac{\lambda - \hat{k}_{\mathrm{E}} - \hat{q}_{\mathrm{E}}}{\lambda^2 + (k_{\mathrm{E}} + q_{\mathrm{E}})^2} \gamma_\mu \frac{\lambda - \hat{k}_{\mathrm{E}}}{\lambda^2 + k_{\mathrm{E}}^2} \gamma_\nu \Bigg\}.
\end{aligned} \tag{269}$$

Again passing to the Minkowski metric and using the generalized Feynman parameterization, one gets,

$$\begin{aligned}
\tilde{\Gamma}_\mu(p_1, p) = \int_{-m}^{m} d\lambda\, \rho(\lambda) \Bigg\{ & -\frac{e^2}{8\pi} \frac{1}{2i} \int_{-\alpha+i\infty}^{-\alpha-i\infty} d\xi \frac{v(\xi)}{(\sin\pi\xi)^2} \\
& \times \frac{(\lambda^2\ell^2)^\xi}{\Gamma(1+\xi)} F_\mu(\xi; p_1, p) \Bigg\},
\end{aligned} \tag{270}$$

where

$$F_\mu(\xi; p_1, p) = \gamma_\mu F_1(\xi; p_1, p) + F_{2\mu}(\xi; p_1, p).$$

Here

$$F_1(\xi; p_1, p) = \frac{1}{\Gamma(1-\xi)} \int_0^1 \int_0^1 \int_0^1 d\alpha\, d\beta\, d\gamma\, \delta(1-\alpha-\beta-\gamma)\alpha^{-\xi} Q^{\xi}$$

and

$$\begin{aligned} F_2(\xi; p_1, p) = \frac{1}{\Gamma(-\xi)} \int_0^1 \int_0^1 \int_0^1 & d\alpha\, d\beta\, d\gamma\, \delta(1-\alpha-\beta-\gamma)\alpha^{-\xi} Q^{\xi-1} \\ & \times \frac{1}{\lambda^2}[\lambda^2\gamma_\mu - 2\lambda q_\mu + 4\lambda(\beta q_\mu - \alpha p_\mu) + (\alpha\hat{p} - \beta\hat{q})\gamma_\mu \hat{q} \\ & + (\alpha\hat{p} - \beta\hat{q})\gamma_\mu(\alpha\hat{p} - \beta\hat{q})]; \end{aligned}$$

$$Q = \beta + \gamma - \alpha\gamma\frac{p^2}{\lambda^2} - \beta\gamma\frac{q^2}{\lambda^2} - \alpha\beta\frac{(p+q)^2}{\lambda^2}. \tag{271}$$

Let us calculate the vertex function (270) for two cases: first, when $q = 0$ and p has an arbitrary value; second, when q is an arbitrary quantity and p and p_1 are situated on the λ-mass shell. In the first case, assuming $q = 0$ in the formula (271) and after some standard calculations, one gets

$$\begin{aligned} F_\mu(\xi; p_1, p) = \frac{1}{\Gamma(1-\xi)} \int_0^1 du \left(\frac{1-u}{u}\right)^{\xi} \left(1 - u\frac{p^2}{\lambda^2}\right)^{\xi} \\ \times \left[u\gamma_\mu + \frac{2\xi u p_\mu(2\lambda - u\hat{p})}{\lambda^2 - up^2}\right]. \end{aligned} \tag{272}$$

Comparing this formula with the expression (266) for the self-energy of the "square-root" electron, it is easily seen that

$$F_\mu(\xi; p, p) = -\frac{\partial}{\partial p_\mu} F(\xi; p) \tag{273}$$

From this identity we can obtain a very important conclusion. In the square-root nonlocal QED constructed using the concept of the extended mass and charge densities, the Ward-Takahashi identity is valid,

$$\tilde{\Gamma}_\mu(p, p) = -\frac{\partial}{\partial p_\mu} \tilde{\Sigma}(p). \tag{274}$$

In the second case, one can put

$$\bar{u}(\mathbf{p_1})\tilde{\Gamma}_\mu(p_1, p)u(\mathbf{p}) = \bar{u}(\mathbf{p}_1)\Lambda_\mu(q)u(\mathbf{p}) \tag{275}$$

where $\bar{u}(\mathbf{p}_1)$ and $u(\mathbf{p})$ are solutions of the Dirac equations

$$(\hat{p} - \lambda)u(\mathbf{p}) = 0 \quad \text{and} \quad \bar{u}(\mathbf{p}_1)(\hat{p}_1 - \lambda) = 0.$$

Substituting the vertex function (270) into (275) and after some transformations, we have

$$\bar{u}(\mathbf{p}_1)F_\mu(\xi; p_1, p)u(\mathbf{p}) = \bar{u}(\mathbf{p}_1)\Lambda_\mu(\xi, q)u(\mathbf{p}). \tag{276}$$

Here

$$\Lambda_\mu(\xi, q) = \gamma_\mu f_1(\xi, q^2) + \frac{i}{2\lambda}\sigma_{\mu\nu}q_\nu f_2(\xi, q^2),$$

$$\sigma_{\mu\nu} = \frac{1}{2i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu),$$

$$f_j(\xi, q^2) = \frac{1}{\Gamma(1-\xi)}\int_0^1\int_0^1\int_0^1 d\alpha\, d\beta\, d\gamma\, \delta(1-\alpha-\beta-\gamma) \times \alpha^{-\xi}L^{\xi-1}g_j(\alpha, \beta, \gamma, q^2),$$

$$L = \xi\alpha + (1-\alpha)^2 - \beta\gamma\frac{q^2}{\lambda^2},$$

$$g_1(\alpha, \beta, \gamma, q^2) = [(1-\alpha)^2(1-\xi) + 2\alpha\xi] - [\beta\gamma + \xi(\alpha+\beta)(\alpha+\gamma)]\frac{q^2}{\lambda^2},$$

and

$$g_2(\alpha, \beta, \gamma, q^2) = 2\alpha(1-\alpha)\xi. \tag{277}$$

To avoid infrared divergences in the vertex function we have introduced here the parameter $\varepsilon = \mu_{\mathrm{ph}}^2/\lambda^2$, taking into account the “mass” of the photon. Finally, one gets

$$\Lambda_\mu(q) = \int_{-m}^{m} d\lambda\, \rho(\lambda)\left[\gamma_\mu F_1(q^2) + \frac{i}{2\lambda}\sigma_{\mu\nu}q_\nu F_2(q^2)\right], \tag{278}$$

where

$$F_j(q^2) = -\frac{e^2}{8\pi}\frac{1}{2i}\int_{\alpha+i\infty}^{-\alpha-i\infty}\frac{d\xi}{(\sin\pi\xi)^2}\frac{v(\xi)}{\Gamma(1+\xi)}(\lambda^2\ell^2)^\xi f_j(\xi, q^2). \tag{279}$$

It is easy to verify that the vertex function $\Lambda_\mu(q)$ satisfies the gauge-invariant condition

$$q_\mu\bar{u}(\mathbf{p}_1)\Lambda_\mu(q)u(\mathbf{p}) = 0. \tag{280}$$

Let us write the first terms of the decomposition for the functions $F_1(q^2)$ and $F_2(q^2)$

over two small parameters $\lambda^2\ell^2$ and q^2/λ^2:

$$F_1(q^2) = \frac{\alpha}{4\pi}\left[\chi - 2\sigma - v'(0) + \frac{9}{2} - 6C - 3\lambda^2\ell^2 v(1)\right] + \frac{\alpha}{2\pi}\frac{q^2}{\lambda^2} \times \left\{\frac{2}{3}\left(\frac{1}{2}\sigma - \frac{3}{8}\right) + \frac{\lambda^2\ell^2}{3}\left[v(1)\left(-\chi + 2C - \frac{13}{6}\right) + v'(1)\right]\right\} \quad (281)$$

where $\sigma = \ln(\lambda^2/\mu_{\mathrm{ph}}^2)$, $C = 0.577215\ldots$ is the Euler constant, $\alpha = e^2/4\pi$, and χ is defined in (268). And

$$F_2(q^2) = -\frac{\alpha}{2\pi}\left[1 - \frac{2}{3}v(1)\lambda^2\ell^2\right]. \quad (282)$$

From this last formula we can see that corrections to the anomalous magnetic moment (AMM) of leptons are given by

$$\Delta\mu = -\int_{-m}^{m} d\lambda\, \rho(\lambda) F_2(q^2) = \frac{\alpha}{2\pi}\left[1 - \frac{1}{3}v(1)m^2\ell^2\right], \quad (283)$$

where we have used the properties (10) for the mass density $\rho(\lambda)$. The first term in (283) corresponds to the Schwinger correction obtained in local QED. From the experimental values of the AMM of the electron and muon one gets the following restriction on the value of the fundamental length ℓ:

$$\ell \leq 1.2 \times 10^{-14}\ \mathrm{cm} \quad \text{for} \quad \Delta\mu_{\mathrm{exp}}^{(\mathrm{e})}$$

and

$$\ell \leq 1.7 \times 10^{-15}\ \mathrm{cm} \quad \text{for} \quad \Delta\mu_{\mathrm{exp}}^{(\mu)}. \quad (284)$$

14.2.4. The Self-Energy of the "Square-Root" Electron

Now the following question arises: how to coordinate classical (256) and quantum (268) expressions for the electron self-energy. As seen here, the quantum contribution (268)

$$\delta m_{\mathrm{qu}} = \int_{-m}^{m} d\lambda\, \rho(\lambda)\left[\frac{3}{16\pi^2}\frac{e^2}{\hbar c}\lambda \ln\left(\frac{\hbar}{\lambda c r_0}\right)^2\right] \quad (285)$$

is even equal to zero after integration over the λ variable. The self-energy of the electron in the classical theory is given by (256)

$$\delta m_{c\ell} = \frac{e^2}{2c^2 r_0}\frac{1}{(2\pi)^{\frac{3}{2}}} \quad (286)$$

Here the parameter $r_0 = \ell$ is the electron radius. Comparing formulas (285) and (286) shows that the quantum correction to the mass value of the electron in difference of the classical one depends on its mass and increases only as the logarithmic function at $r_0 \to 0$. The passage $\hbar \to 0$ in (285) that corresponds to the classical limit leads to an explicitly senseless result; therefore formulas (285) and (286) say that the corresponding principle between the quantum and classical theories is not valid in the problem of the self-energy of the electron. However, in our scheme one can expect that the corresponding principle will satisfy exactly. Indeed, it turns out that in the second order of the perturbation theory quantum corrections, δm_{qu}, goes to classical ones, $\delta m_{\mathrm{c}\ell}$, at the limit $\hbar \to 0$, and all higher order contributions turn to zero at $\hbar \to 0$. The proof of this assertation is the same as in the Efimov (1977) nonlocal theory.

Let us consider in detail the structure of the integral defining the corrections to the electron mass. For this purpose we turn to the expression (264) and use the standard definition

$$\delta\lambda = -\lambda[A(\lambda^2) + B(\lambda^2)]$$

and

$$Z_2 - 1 = B(\lambda^2) + 2\lambda^2[A'(\lambda^2) + B'(\lambda^2)],$$

where $\delta\lambda$ and Z_2 are renormalization constants in the mass operator

$$\Sigma_r(p) = \{\lambda A(p^2) + B(p^2)\hat{p}\} + \delta\lambda - (Z_2 - 1)(\hat{p} - \lambda),$$

and choose the system of reference, where the vector $p_{\mathrm{E}} = (-i\lambda, 0)$, one gets

$$\delta m = \int_{-m}^{m} d\lambda\, \rho(\lambda) \frac{e^2}{(2\pi)^4 \hbar} \int \frac{d^4k_{\mathrm{E}}\; V\left(\frac{k_{\mathrm{E}}^2 \ell^2}{\hbar^2}\right)}{k_{\mathrm{E}}^2} \cdot \frac{2\lambda - 2ik_4}{k_{\mathrm{E}}^2 - 2i\lambda k_4}. \tag{287}$$

Here we have shown an explicit dependence on the Planck constant $\hbar$ since we will be interested in the limit $\hbar \to 0$. Integrating over the Euclidean angles and some transformations later, the expression (287) takes the form

$$\delta m = \int_{-m}^{m} d\lambda\, \rho(\lambda) \frac{\lambda}{(2\pi)^2} \frac{e^2}{\hbar} \int_0^{\infty} du\, V\left(4\left(\frac{\ell}{\lambda_0}\right)^2 u\right) Q(u) \tag{288}$$

where $Q(u) = 2u + (1 - 2u)\sqrt{1 + 1/u}$, and $\lambda_0 = \hbar/\lambda c$ is the Compton wavelength of the electron with the "stochastic" mass λ. From the formulas obtained we see that contributions to the mass δm have two limits (285) and (286) for the cases $\ell \ll \lambda_0$ and $\ell \gg \lambda_0$, respectively. Indeed, asymptotic behaviour of the function $Q(u)$ is given by

$$Q(u) = \begin{cases} 1/\sqrt{u}, & u \ll 1; \\ 3/4u, & u \gg 1. \end{cases}$$

On the other hand the function $V(u) = \exp[-u/4]$ is the order of unit for $u \leq 1$

and decreases rapidly for $u > 1$, such that

$$\delta m \simeq \int_{-m}^{m} d\lambda\, \rho(\lambda)\lambda \frac{e^2}{\hbar c} \int_0^{(\frac{\lambda_0}{\ell})^2} du\, Q(u).$$

For $\ell \ll \lambda_0$ contribution to the integrand is given by large quantities of u; therefore,

$$\int_0^{(\frac{\lambda_0}{\ell})^2} du\, Q(u) \simeq \frac{3}{4} \int_1^{(\frac{\lambda_0}{\ell})^2} \frac{du}{u} = \frac{3}{4} \ln\left(\frac{\lambda_0}{\ell}\right)^2$$

and

$$\delta m \simeq \int_{-m}^{m} d\lambda\, \rho(\lambda) \frac{3\lambda}{16\pi^2} \frac{e^2}{\hbar c} \ln\left(\frac{\lambda_0}{\ell}\right)^2 = 0, \tag{289}$$

i.e., we obtain the quantum field expression for the self-mass (285). For $\ell \gg \lambda_0$ contribution to the integrand is defined by small values of u, and therefore

$$\int_0^{(\frac{\lambda_0}{\ell})^2} du\, Q(u) \simeq \int_0^{(\frac{\lambda_0}{\ell})^2} \frac{du}{\sqrt{u}} = \frac{2\lambda_0}{\ell}$$

and

$$\delta m \simeq \int_{-m}^{m} d\lambda\, \rho(\lambda) \cdot \lambda \frac{e^2}{\hbar c} \cdot \text{const}\, \frac{\hbar}{\lambda \cdot c \cdot \ell} = \frac{1}{(2\pi)^{\frac{3}{2}}} \cdot \frac{e^2}{2c^2 \ell},$$

i.e., we get the classical expression for the self-mass (286), which does not depend on the Planck constant or the electron mass m.

Thus, formulas obtained for the corrections to the mass (285) and (286) are valid for any relations between the fundamental length ℓ and the Compton wave length of the electron $\lambda_0 = \hbar/\lambda c$, and for the limits $\ell \ll \lambda_0$ and $\ell \gg \lambda_0$ give true asymptotic value for quantum and classical theories, respectively.

Finally, it should be noted that as shown in Efimov (1977) contributions from all higher orders of the perturbation theory turn out to be zero at the limit $\hbar \to 0$. This assertion is valid in our case. Moreover, the equality (289) holds for the full massive operator $\tilde{\Sigma}(p)$ on the λ-mass shell:

$$\tilde{\Sigma}(p) = \frac{e^2}{(2\pi)^4 \hbar c} \int d^4k\, \gamma_\mu G(p-k)\Gamma_\mu(p-k, p)\Delta(k) \tag{290}$$

where $\lambda^2 = p^2$, $\hat{p}u(\mathbf{p}) = \lambda u(\mathbf{p})$, G and Δ are full Green functions of the electron with the stochastic mass λ and photon, and Γ is full vertex part. We know that Green functions of the electron and photon decrease as $G(k) = O(1/k)$ and $\Delta(k) = O(1/k^2)$ for $k \to \infty$. In accordance with the Ward identity,

$$\Gamma_\mu(p-k, p) = O\left(\frac{1}{k}\right).$$

From this we conclude that expression (290), like (285), diverges as a logarithmic function and is proportional to λ and therefore

$$\delta m_{\rm qu}^{\rm full} = -\int_{-m}^{m} d\lambda\, \rho(\lambda)\Sigma(p^2 = \lambda^2) = 0$$

for all orders of the perturbation theory.

In conclusion we notice that in the square-root quantum electrodynamics it is not necessary to regularize the square-root electron mass, i.e., the physical mass of the electron, m, coincides with the bare one, m_o, in the Lagrangian (225).

15. EXTENDED CHARGES AND SUPERFIELD

Recent rapid and exciting developments in the theories of spread-out (extended) or nonlocal objects like strings and D and p branes, and in their duality properties, need a practical working model allowing to shed light on those low-energy limits by means of experimental (particle) physics. In this section we consider extended electric charges like charged ring, disk, holed coin, and the torus in an attempt to understand an essential difference between these structures in the point of view of the supersymmetric string theory (Bailin and Love, 1994). It turns out that all considered innermost structures of charge: except the torus, all are associated with simple nonlocal (Efimov, 1977, 1985; Namsrai, 1986) or bilocal (Yukawa, 1950a,b,c) theories with formfactors, while the charged torus gives rise to the appearance of a new kind nonlocal theory, wherein boson and fermion fields are generated simultaneously. Roughly speaking, the charged tori radiates or absorbs at the same time photon and neutrino-like massless fermion.

Our construction of the theory of extended charges is based on the Feynman–Schwinger–Yukawa correspondence rule, which asserts that in the static limit, potentials between two sources (interacting charges) are associated to the Fourier transform of propagators of force-transmitting particles in the momentum space and vice versa. For example, see formulas (51)–(54), (205), (206), and Section 11.

The next step is to understand the physical meaning of square-root (Weyl, 1927) propagators $(-p^2 - i\varepsilon)^{-1/2}$ and $(m^2 - p^2 - i\varepsilon)^{-1/2}$, $(p^2 = p_0^2 - \mathbf{p}^2)$, which appeared in the charged and massive torus cases.

As seen below, solutions (15) and (22), and (13) and (21) with the properties (10) allow us to shed light on the torus field by means of the square-root-operator formalism. In other words, the fermion partner of the boson field, which are simultaneously generated by the torus field, is not the usual Dirac spinor but a new kind of spinor field with random mass distribution over the torus. In the massless limit $m \to 0$ they become the photons (gravitons) and neutrino-like particles, the former being carriers of electromagnetic (or gravitational) interactions. Notice that due to the distribution $\rho(\lambda)$ (10) in most of the cases, integration over the variable λ can be done onto an unit circle. This property allows us to present torus

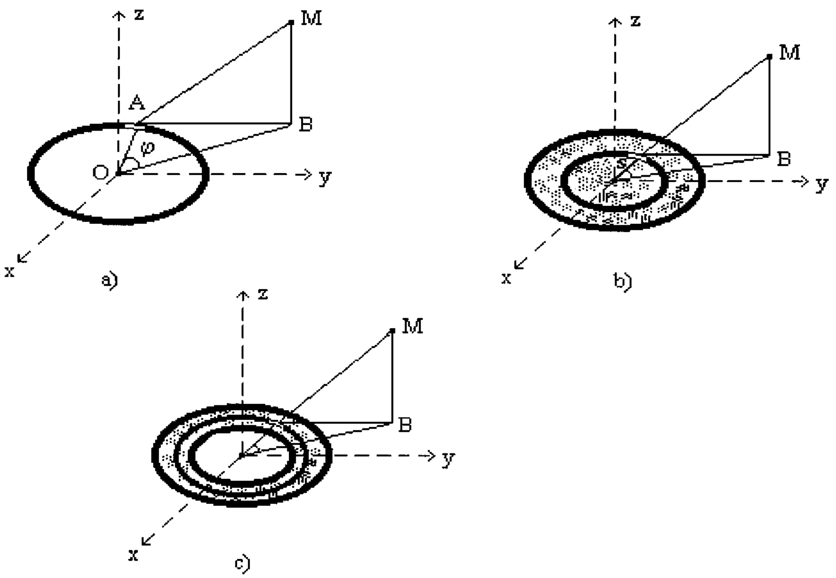

Fig. 2. (a) One-dimensional extended charge of the ring. (b) Two-dimensional extended charge of the disk. (c) Two-dimensional extended charge of the holed coin.

fields (strings) onto a circle winding around the torus, indicating the importance of square-root-operator formalism in torus physics.

15.1. One- and Two-Dimensional Extended Charge

15.1.1. Charge of the Ring

Let us consider the ring charge (Fig. 2(a)) and calculate its potential. Here the electric charge e is distributed uniformly with a linear density σ on a circle of the radii ℓ. The charge element de corresponding to a linear differential length ds of the ring is given by $de = \sigma ds$. Since $ds = \ell d\varphi$, we get $e = 2\pi\ell\sigma$. The potential at the point M, generated by the element ds of the ring is

$$dU_{\mathrm{r}}(M) = \frac{1}{4\pi}\frac{de}{\sqrt{z^2 + (\ell^2 + \rho^2 - 2\ell\rho\cos\varphi)}}, \tag{291}$$

where we have used the cylindrical system of co-ordinates and the cosine's theorem for triangle OAB. After some elementary integrations we obtain the potential of the ring in the form

$$U_{\mathrm{r}}(z,\rho) = \frac{e}{4\pi\sqrt{z^2 + (\rho+\ell)^2}}\frac{2}{\pi}\cdot F\left(\frac{\pi}{2}, \frac{2\sqrt{\ell\rho}}{\sqrt{z^2 + (\rho+\ell)^2}}\right), \tag{292}$$

where $F(\frac{\pi}{2}, k)$ is the complete elliptic integral of the first kind,

$$K(k) = F\left(\frac{\pi}{2}, k\right) = \frac{\pi}{2} N_n \cdot k^{2n} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right). \tag{293}$$

Here $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function and

$$k = \frac{2\sqrt{\ell\rho}}{\sqrt{z^2 + (\rho + \ell)^2}}, \quad N_n = 1 + \sum_{n=1}^{\infty} \left[\frac{(2n-1)!!}{2^n \cdot n!}\right]^2. \tag{294}$$

For the case $\rho = 0$ we have the well-known textbook formula

$$U_{\rm r}(z) = \frac{e}{4\pi\sqrt{z^2 + \ell^2}}.$$

Notice that $U_{\rm r}(0) = e/4\pi\ell$ is finite at the origin.

Now a question arises: which kind of a field does the ring charge radiate or absorb compared to the pointlike Coulomb charge? We use the standard method as mentioned previously. Let us consider a radiation field in the direction of the z axis, where momentum of the particle has the components $\mathbf{p} = (p_z, 0, 0) = (p, 0, 0)$, so that the Green function associated with the ring potential is given by

$$\tilde{D}_{\rm r}(\mathbf{p}) = \int d^3r\, e^{-i\mathbf{pr}}(U_r(r)/e), \tag{295}$$

where the potential $U_{\rm r}(r)$ is expressed by the formula (292). Taking into account the series (293) and using the cylindrical system of co-ordinates one can represent this function in the form

$$\tilde{D}_{\rm r}(\mathbf{p}) = N_n \cdot D_{\rm r}^{\rm n}(p), \quad p = \sqrt{p^2}. \tag{296}$$

Here

$$\tilde{D}_{\rm r}^{\rm n}(p) = 4^n \ell^n \int_0^\infty d\rho \cdot \rho^{n+1} \int_0^\infty dz\, \cos pz[z^2 + (\rho + \ell)^2]^{-n-\frac{1}{2}} \tag{297}$$

and notation N_n is used from (294). The last integral is calculated explicitly.

$$\begin{aligned} I &= \int_0^\infty dz \cos pz[z^2 + (\rho + \ell)^2]^{-n-\frac{1}{2}} \\ &= \frac{1}{\sqrt{\pi}}\left(\frac{2(\ell + \rho)}{p}\right)^{-n} \cos \pi n \cdot \Gamma\left(\frac{1}{2} - n\right) K_n(p(\rho + \ell)). \end{aligned}$$

Now let us use the Mellin representation for the Bessel function $K_\nu(z)$ of the imaginary argument

$$K_\nu(z) = \frac{\sqrt{\pi}}{2^{\nu+1}} \frac{1}{2i} \int_{-\beta+i\infty}^{-\beta-i\infty} \frac{d\xi}{\sin \pi\xi} \frac{1}{\Gamma(1+\xi)} \frac{\Gamma\left(-\nu - \frac{\xi}{2}\right)}{\Gamma\left(\frac{1-\xi}{2}\right)} z^{\nu+\xi}, \tag{298}$$

where $2\nu < -\beta$. So that expression (297) takes the form

$$\tilde{D}_{\rm r}^{\rm n}(p) = \pi \frac{\Gamma(n+2)}{\Gamma\left(\frac{1}{2}+n\right)} \ell^2 M_\xi \cdot (p^2\ell^2)^{\xi-1}, \tag{299}$$

where we have used short notation

$$M_\xi = \frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty} \frac{d\xi}{\sin^2 \pi\xi} \frac{1}{(1-2\xi+2n)} \frac{\Gamma\left(\frac{1}{2}+\xi-n\right)}{\Gamma(\xi)\Gamma(1+2\xi-n)} \quad (-1 \le \beta < 0). \tag{300}$$

Finally, in the four-dimensional momentum space the Green function of the nonlocal photon field generated by the ring charge acquires the form

$$\tilde{D}_{\rm r}(p) = \frac{1}{-p^2 - i\varepsilon} V_{\rm r}(-p^2\ell^2), \quad p^2 = p_0^2 - \mathbf{p}^2, \tag{301}$$

where

$$V_{\rm r}(-p^2\ell^2) = \pi N_n \frac{\Gamma(n+2)}{\Gamma\left(n+\frac{1}{2}\right)} \tilde{D}_{\rm r}^{\rm n}(p) \tag{302}$$

and

$$\tilde{D}_{\rm r}^{\rm n}(p) = M_\xi[-p^2\ell^2]^\xi. \tag{303}$$

Here N_n and M_ξ are given by (294) and (300). Further, we use the Mellin representation (303), displace its integration contour to the right, and take corresponding residues. Then the resulting form factor $V_{\rm r}(-p^2\ell^2)$ converges very well. For example, sum of the first 11 terms in (299) is

$$\tilde{D}_{\rm r}(p) = \frac{1}{-p^2 - i\varepsilon} - \frac{23}{5\cdot 4\cdot 11}\ell^2 + O(p^2\ell^4). \tag{304}$$

Thus, we have arrived at the Efimov (1977) nonlocal theory with form factor (302) (see, also, Namsrai, 1986) for the ring charge case.

15.1.2. Charge of the Disc

In this case, the surface element of charge defines as $de = \lambda ds \cdot s d\varphi$, where λ is a surface charge density. Here thin ring of radius s and width ds on the disc with radii ℓ is considered (Fig. 2b). For the disc charge, expression (292) for the potential acquires the form

$$U_{\rm d}(M) = \frac{\lambda}{4\pi}\int_0^\ell ds \cdot s \int_0^{2\pi} d\varphi \frac{1}{\sqrt{z^2+s^2+\rho^2-2s\rho\cos\varphi}}. \tag{305}$$

After similar and elementary calculations as shown previously one gets

$$\tilde{D}_{\rm d}(p) = \frac{1}{-p^2-i\varepsilon} V_{\rm d}(-p^2\ell^2) \sim \frac{1}{-p^2-i\varepsilon} - \frac{23\ell^2}{10\cdot 4\cdot 11} + O(p^2\ell^4), \tag{306}$$

where

$$V_{\mathrm{d}}(-p^2\ell^2) = \pi N_{\mathrm{n}} \cdot \frac{\Gamma(n+2)}{\Gamma\left(n+\frac{1}{2}\right)} \tilde{D}^{\mathrm{n}}_{\mathrm{d}}(p) \tag{307}$$

and

$$\tilde{D}^{\mathrm{n}}_{\mathrm{d}}(p) = M_\xi \cdot \frac{1}{1+\xi}[-p^2\ell^2]^\xi \tag{308}$$

These formulas say that two-dimensional charged disc radiates or absorbs nonlocal photon fields, no other physical characteristics exist here.

15.1.3. Charge of the Holed Thin Coin

In this case potential (305) is reduced to the form (Fig. 2c)

$$U_{\mathrm{hc}} = \frac{\lambda}{4\pi}\int_r^R ds \cdot s \int_0^{2\pi} d\varphi \frac{1}{\sqrt{z^2+s^2+\rho^2-2\rho s\cos\varphi}}, \tag{309}$$

where R and r are big and small radius of the holed coin. Propagator of the nonlocal photon generated by the charged holed thin coin is given by

$$\tilde{D}_{\mathrm{hc}}(p) = \frac{1}{-p^2-i\varepsilon} V_{\mathrm{hc}}(-p^2R^2). \tag{310}$$

Here

$$V_{\mathrm{hc}}(-p^2R^2) = \pi N_n \cdot \frac{\Gamma(n+2)}{\Gamma\left(n+\frac{1}{2}\right)} \tilde{D}^{\mathrm{n}}_{\mathrm{hc}}(p) \tag{311}$$

and

$$\tilde{D}^{\mathrm{n}}_{\mathrm{hc}}(p) = \frac{R^2}{R^2-r^2} M_\xi \cdot \frac{1}{1+\xi}[-R^2p^2]^\xi \left[1-\left(\frac{r}{R}\right)^{2\xi+2}\right], \tag{312}$$

where N_n and M_ξ are given by (294) and (300), respectively.

Thus we obtain similar results as in the ring and disc charges cases, as it should.

15.2. Three-Dimensional Extended Charge

15.2.1. Charge of the Torus and Its Potential

Now we turn to study a very interesting three-dimensional object, where the electric charge is distributed uniformly on the surface of the torus.

At first sight one may think that the torus has a complicated geometrical structure to set up its potential. However, if we look at it carefully we can observe its nice geometric construction, having common characteristics with the ring, except for an extra one-degree-of-freedom winding around the torus.

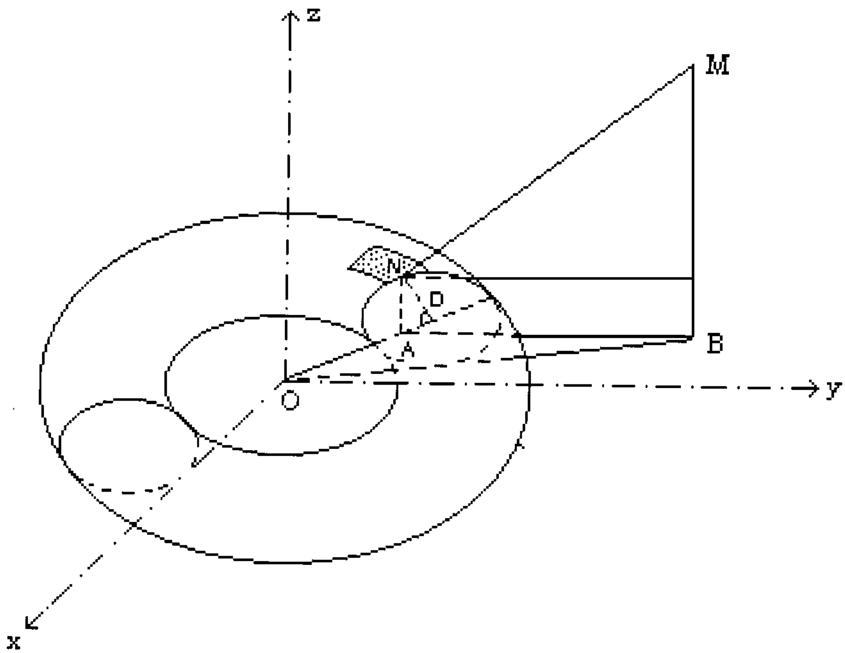

Fig. 3. Three-dimensional extended charge of the torus.

Let us consider the surface differential element (protruberant trapezium) ds on the torus, the centre of which belongs to the point N (Fig. 3). Further, we trace two lines from the point N: AN $= h$ is perpendicular to the plane OXY and ND crosses with the central line of the torus, i.e. ND $= (R-r)/2$. An angle $<$ADN is denoted by α, $0 \le \alpha \le 2\pi$. Thus, by construction

$$\mathrm{NM}^2 = (z-h)^2 + \mathrm{AB}^2, \tag{313}$$

where

$$\begin{cases} h = \dfrac{R-r}{2}\sin\alpha, & \mathrm{AB}^2 = \rho^2 + L^2 - 2\rho L\cos\varphi, \\ L = r + \dfrac{R-r}{2}(1-\cos\alpha) & \end{cases} \tag{314}$$

Since the medium line of the trapezium ds is ON $\cdot\, d\varphi = \sqrt{h^2+L^2}\, d\varphi$, its area is

$$ds = \left(\frac{R-r}{2}\right) d\alpha \sqrt{\left(r + \frac{R-r}{2}(1-\cos\alpha)\right)^2 + \left(\frac{R-r}{2}\right)^2 \sin^2\alpha} \cdot d\varphi. \tag{315}$$

Thus, the element of charge corresponding to the surface differential form ds of the torus is defined as

$$de = \lambda\, ds = \lambda\left(\frac{R-r}{2}\right)\sqrt{L^2+h^2}\, d\alpha\, d\varphi = \frac{R-r}{2}\cdot\lambda\cdot r\sqrt{1+q^2\,\sin^2\frac{\alpha}{2}}\, d\alpha\, d\varphi \tag{316}$$

where $q^2 = R^2/r^2 - 1$, λ is the surface charge density, φ is the polar angle ($0 \leq \varphi \leq 2\pi$), and α is the winding angle around the torus. The potential element $dU_{\mathrm{T}}(\mathrm{M})$ generated by the charge de of the torus at the point M given by

$$dU_{\mathrm{T}}(\mathrm{M}) = \frac{de}{4\pi\sqrt{\mathrm{NM}^2}}, \tag{317}$$

where de and NM^2 are defined by expressions (316) and (313).

Integration over the polar angle $d\varphi$ (as was done in the ring charge case) gives

$$U_{\mathrm{T}}(z,\rho) = \pi\lambda(R-r)\int_0^{2\pi} d\alpha\,\sqrt{L^2+h^2}\,\frac{1}{4\pi\sqrt{(z-h)^2+(\rho+L)^2}} \times \frac{2}{\pi}F\left(\frac{\pi}{2}, \frac{2\sqrt{L\rho}}{\sqrt{(z-h)^2+(\rho+L)^2}}\right), \tag{318}$$

where parameters L and h are given by (314). We see that potential form (318) is very similar to the ring charge one. This allows us to integrate all results obtained in the torus case up to the end. Let us calculate the surface of the torus. From (316) it follows

$$S_{\mathrm{t}} = \oint_{\Sigma} ds = 4\pi R(R-r)E\left(\frac{\pi}{2}, \sqrt{1-\frac{r^2}{R^2}}\right).$$

The torus potential (318) is also finite at the origin

$$U_{\mathrm{T}}(0) = \frac{e}{8RE\left(\frac{\pi}{2}, \sqrt{1-\frac{r^2}{R^2}}\right)},$$

where $E(\pi/2, k)$ is the complete elliptic integral of the second kind.

15.2.2. The Green Function Generated by Charge of Torus

According to the general rule expounding in previous sections, the Green function caused by the torus charge is given by

$$\tilde{D}_{\mathrm{t}}(p) = \int d^3r\, e^{-i\mathbf{pr}}(U_{\mathrm{T}}/e_{\mathrm{t}}) \tag{319}$$

Making use of formulas (315) and (316), one can easily calculate the full charge of the torus

$$e_{\rm t} = \lambda\left(\frac{R-r}{2}\right)\int_0^{2\pi} d\varphi\int_0^{2\pi} d\alpha\,\sqrt{L^2+h^2} = 4\pi R(R-r)\lambda E\left(\frac{\pi}{2},\sqrt{1-\frac{r^2}{R^2}}\right). \tag{320}$$

Now we use the cylindrical system of co-ordinates and carry out some calculations as in the ring charge case. The result reads

$$\tilde{D}_{\rm t}(p) = \pi N_n\cdot\frac{\Gamma(n+2)}{\Gamma\left(n+\frac{1}{2}\right)}\tilde{D}^{\rm n}_{\rm t}(p), \tag{321}$$

where

$$\tilde{D}^{\rm n}_{\rm t}(p) = \frac{2\pi\lambda}{e_{\rm t}}\left(\frac{R-r}{2}\right)\int_0^{2\pi} d\alpha\,\sqrt{L^2+h^2}\,e^{iph}L^2M_\xi(-p^2L^2)^{\xi-1}. \tag{322}$$

Here definitions of N_n and M_ξ are the same as in (294) and (300).

Making use of the Mellin representation

$$e^{iph} = \frac{1}{2i}\int_{-\gamma+i\infty}^{-\gamma-i\infty}\frac{d\beta}{\sin\pi\beta}\frac{(ph)^{2\beta}}{\Gamma(1+2\beta)}\left[1+iph\frac{1}{1+2\beta}\right],\quad (-1<\gamma<0) \tag{323}$$

one can see that propagator (321) consists of two different parts

$$\tilde{D}_{\rm t}(p) = \tilde{D}^{\rm ph}_{\rm t}(p)+\tilde{D}^{\rm f}_{\rm t}(p), \tag{324}$$

where

$$\tilde{D}^{\rm ph}_{\rm t}(p) = \frac{1}{-p^2-i\varepsilon}V^{\rm ph}_{\rm t}(-p^2R^2),\qquad \tilde{D}^{\rm f}_{\rm t}(p) = \frac{i}{\sqrt{-p^2-i\varepsilon}}V^{\rm f}_{\rm t}(-p^2R^2). \tag{325}$$

Here

$$V^{\rm ph}_{\rm t}(-p^2R^2) = \pi N_n\cdot\frac{\Gamma(n+2)}{\Gamma\left(n+\frac{1}{2}\right)}\tilde{D}^{\rm ph}_{\rm nt}(p), \tag{326}$$

and

$$V^{\rm f}_{\rm t}(-p^2R^2) = \pi N_n\cdot\frac{\Gamma(n+2)}{\Gamma\left(n+\frac{1}{2}\right)}\tilde{D}^{\rm f}_{\rm nt}(p). \tag{327}$$

Following expressions hold for functions $\tilde{D}^{\rm ph}_{\rm nt}(p)$ and $\tilde{D}^{\rm f}_{\rm nt}(p)$:

$$\tilde{D}^{\rm ph}_{\rm nt}(p) = \frac{r}{4\pi E}\int_0^{2\pi} d\alpha\sqrt{1+q^2\sin^2\frac{\alpha}{2}}\,\frac{1}{2i}\times\int_{-\gamma+i\infty}^{-\gamma-i\infty}\frac{d\beta\left(\frac{h}{L}\right)^{2\beta}}{\sin\pi\beta\,\Gamma(1+2\beta)}M_\xi(-p^2L^2)^{\beta+\xi} \tag{328}$$

and

$$\tilde{D}^{\mathrm{f}}_{\mathrm{nt}}(p) = \frac{r}{4\pi E}\int_0^{2\pi} d\alpha\sqrt{1+q^2\sin^2\frac{\alpha}{2}}\cdot h\cdot\frac{1}{2i} \times \int_{-\gamma+i\infty}^{-\gamma-i\infty}\frac{d\beta\left(\frac{h}{L}\right)^{2\beta}}{\sin\pi\beta\,\Gamma(2+2\beta)}M_{\xi}(-p^2L^2)^{\beta+\xi} \tag{329}$$

where $q^2 = \frac{R^2}{r^2} - 1$; E is the complete elliptic integral of the second kind.

Assuming $p^2R^2 \ll 1$ one can calculate integrals (328) and (329). Displacing contours in (328) and (329) to the right and taking residues, and after some elementary integration over the angle α one gets

$$V_{\mathrm{t}}^{\mathrm{ph}}(-p^2R^2) = 1 + \frac{r^2p^2}{5\cdot 4\cdot 11}\left\{23 + \frac{46}{3}\frac{r}{R+r}\frac{1}{E}\left(2\frac{R^2}{r^2}E - F - E\right) + \frac{R^2}{15E(R+r)^2}\left[2\left(14\frac{r^2}{R^2} - 101\right)F + \frac{R^2}{r^2}E \times\left(404 - 519\frac{r^2}{R^2} + 289\frac{r^4}{R^4}\right)\right]\right\} + O(p^4R^2) \tag{330}$$

and

$$V_{\mathrm{t}}^{\mathrm{f}}(-p^2R^2) = \frac{rR}{3(R+r)}\frac{1}{E}\left(\frac{R}{r} - \frac{r^2}{R^2}\right) + O(p^2R^3). \tag{331}$$

More shortly,

$$V_{\mathrm{t}}^{\mathrm{ph}}(-p^2R^2) \sim 1 + \frac{101}{5\cdot 11\cdot 15}R^2p^2$$
$$V_{\mathrm{t}}^{\mathrm{f}}(-p^2R^2) \sim \frac{2}{3\pi}R \tag{332}$$

Functions $E(\frac{\pi}{2}, \sqrt{1-\frac{r^2}{R^2}})$ and $F(\frac{\pi}{2}, \sqrt{1-\frac{r^2}{R^2}})$ in (330) and (331) are complete elliptic integrals of the second and the first kinds, respectively. It should be noted that it is impossible to explain formulas (324) and (325) as one particle exchange between interactions of the torus charges. This problem will be discussed in Section 15.4.

Generalization of formulas (324)–(329) for the massive case is not difficult. By the common rule we can change $-p^2 \to m^2 - p^2$ in these formulas and obtain the nonlocal Klein–Gordon propagator and its square-root version.

15.3. The Square-Root Green Function and Physical Interpretation

From expression (325) we see that in torus physics the square-root Green function appears. That is,

$$\frac{i}{\sqrt{-p^2 - i\varepsilon}} = \lim_{m\to 0} \frac{i}{\sqrt{m^2 - p^2 - i\varepsilon}} = \lim_{m\to 0} \frac{i}{\pi}\int_{-1}^{1} \frac{d\lambda}{\sqrt{1-\lambda^2}} \frac{\hat{p} + m\lambda}{\lambda^2 m^2 - p^2 - i\varepsilon}, \tag{333}$$

where $\hat{p} = \gamma^\mu p^\mu; \gamma^\nu$ are the Dirac γ matrices. Since

$$\frac{1}{\pi}\int_{-1}^{1} \frac{d\lambda}{\sqrt{1-\lambda^2}} = 1$$

we obtain the propagator of a massless fermion in the limit $m \to 0$ as

$$\tilde{\Delta}_{\mathrm{t}}^{f}(p) = i\frac{\hat{p}}{-p^2 - i\varepsilon}. \tag{334}$$

For x space, first term in (324) gives the nonlocal photon propagator

$$D_{\mu\nu}^{\mathrm{ph}}(x) = ig_{\mu\nu}\frac{1}{(2\pi)^4}\int d^4p\, e^{ipx}\tilde{D}_{\mathrm{t}}^{\mathrm{ph}}(p).$$

Now let us consider the case when the mass of a particle is distributed uniformly on the torus: Its full mass is

$$m_{\mathrm{t}} = \oint dm_{\mathrm{t}} = 4\pi R(R-r)\chi E\left(\frac{\pi}{2}, \sqrt{1-\frac{r^2}{R^2}}\right), \tag{335}$$

where χ is the surface density of a particle mass. On the other hand, equation of motion for some unknown particle and its Green function satisfy Eqs. (7) and (5), and its energy is given by

$$E = \int_{-1}^{1} \rho(\lambda)\, d\lambda \sqrt{\mathbf{p}^2 + m^2\lambda^2} = \frac{2}{\pi}\sqrt{m^2 + \mathbf{p}^2}E\left(\frac{\pi}{2}, \frac{m}{\sqrt{m^2+\mathbf{p}^2}}\right) \tag{336}$$

or

$$E = E_{\mathrm{Ein}} \cdot N_n' \cdot k^{2n}, \tag{337}$$

where $E_{\mathrm{Ein}} = \sqrt{m^2 + \mathbf{p}^2}$, $k = m/E_{\mathrm{Ein}}$, and

$$N_n' = 1 - \sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^n \cdot n!}\right]^2 \frac{1}{2n-1}.$$

We see that for square-root particles the Einstein or relativistic formula for the particle's energy is changes little relative to the formulas (336) and (337). For ultrarelativistic particles this difference becomes smaller and smaller.

This fact tells us that in the processes of radiation of particles by the charge of the torus, the energy balance may be lost but it does not take place. From the

Einstein formula for energy of a particle, energy of the torus is

$$E_{\mathrm{t}} = m_{\mathrm{t}} \frac{c^2}{\sqrt{1 - \frac{v_{\mathrm{t}}^2}{c^2}}} = \frac{4\pi R(R-r)\chi c^2}{\sqrt{1 - \frac{v_{\mathrm{t}}^2}{c^2}}} E\left(\frac{\pi}{2}, \sqrt{1 - \frac{r^2}{R^2}}\right) \tag{338}$$

where $\mathbf{v}_{\mathrm{t}}$ is the torus velocity. To compensate energy loss in the processes of radiation the torus could change its innermost structure, i.e., parameters r and R defining its structure become changeable quantities in accordance with formula (336). For example, let r run between 0 and R, then the case $r = 0$ gives the rest energy of the torus, namely

$$m_{\mathrm{t}}^0 = 4\pi R^2 \chi, \tag{339}$$

so that formula (338) takes the form

$$E_{\mathrm{t}} = \left(\frac{R-r}{R}\right) \frac{E\left(\frac{\pi}{2}, \sqrt{1 - \frac{r^2}{R^2}}\right)}{\sqrt{1 - v_{\mathrm{t}}^2/c^2}} m_{\mathrm{t}}^0 c^2, \quad 0 \le r \le R, \tag{340}$$

which is co-ordinated with formula (336), as it should. In other words, the torus is able to gain energy from vacuum by changing its innermost structure defined by parameters r and R. The changing structure of the torus may be understood in terms of the Lorentz contraction. Indeed, at rest the torus has maximum surface (i.e., surface of the sphere (339), which corresponds to $r = 0$) and the hole in it has not appeared. When the torus moves, its transverse size begins to contract, and a hole starts to form in the torus and its surface becomes smaller and smaller. On the other hand, when the torus turns to move, its energy is increased due to the Lorentz factor $\sqrt{1 - v_{\mathrm{t}}^2/c^2}$ in (340). Thus, we observe an important fact that energy of a particle (square-root) depends, at the same time, on its velocity and on its size (structure).

In the limiting case $v_{\mathrm{t}} = c$ surface of the torus tends to zero (i.e., $r = R$), as does the Lorentz factor, simultaneously. As a result a particle ring acquires a finite-energy value, traveling at the speed of light, as it should. Notice that in this limiting case the torus is converted to the ring with radii R. On the contrary, when the torus loses its energy in processes of radiations its speed decreases and at the same time its surface increases. As a result, the torus is able to gain energy from vacuum. Further, the torus with gained energy is again radiated and so on. This chain of processes is repeated infinite times—at least up to the time when the torus goes to rest.

All these statements can be expressed by a single mathematical formula. Indeed, comparing the two formulas (336) and (340)

$$\frac{2}{\pi}\sqrt{m^2 + \mathbf{p}^2} E\left(\frac{\pi}{2}, \frac{m}{\sqrt{m^2 + \mathbf{p}^2}}\right) \sim \frac{R-r}{R} \frac{m}{\sqrt{1-\beta^2}} E\left(\frac{\pi}{2}, \sqrt{1 - \frac{r^2}{R^2}}\right)$$

and assuming

$$\sqrt{m^2+\mathbf{p}^2} = m(1+\beta^2)^{\frac{1}{2}} = \frac{R-r}{R} m(1-\beta^2)^{-\frac{1}{2}},$$

one obtains

$$\beta^2 = \sqrt{1-\left(\frac{R-r}{R}\right)^2} = \begin{cases} 0 & \text{for} \quad r=0, \\ 1 & \text{for} \quad r=R, \end{cases} \tag{341}$$

as it should. Here $\mathbf{p} = m\mathbf{v}_\mathrm{t}$, and $\beta^2 = v_\mathrm{t}^2/c^2$.

This deeper connection between gain of energy by changing structure of a particle may be also understood in terms of fluctuation in space–time points around the torus. We suppose that the motion of the torus satisfies the square-root-operator equation like (7). Then for its Green function there exists another representation

$$\begin{aligned} S(x) &= \int_{-1}^{1} \rho(\lambda)\, d\lambda\, \frac{1}{(2\pi)^4}\frac{1}{i} \int d^4p\, e^{ipx}\, \frac{\hat{p}+m\lambda}{m^2\lambda^2-p^2-i\varepsilon} \\ &= \int_{-1}^{1} d\lambda\, \lambda^3 \rho(\lambda)\, S^{\mathrm{c}}\,(x\lambda, m), \end{aligned} \tag{342}$$

and for the wave function

$$\varphi(x) = \int_{-1}^{1} d\lambda\, \lambda^{\frac{3}{2}} \rho(\lambda)\psi(x\lambda, m). \tag{343}$$

Here $\psi(x\lambda, m)$ and $S^\mathrm{c}\ (x\lambda, m)$ are the Dirac spinor of mass m and its Green function but space–time points, $x\lambda = x_0\lambda$, xλ, have the random distribution $\rho(\lambda)$ with different weights λ^3 and $\lambda^{3/2}$, depending on the physical characteristics of the torus.

Finally, after integration of the variable λ one can represent the Green function: its the Euclidean version and the plane wave solution of a square-root particle in the following forms

$$S(x) = \frac{3}{8}\frac{1}{(2\pi)^4} \int d^4p \frac{m+\hat{p}}{m^2-p^2-i\varepsilon}(px)_1F_2\left(\frac{5}{2};\frac{3}{2},3;-\frac{(px)^2}{4}\right), \tag{344}$$

$$\begin{aligned} S_\mathrm{E}(x_\mathrm{E}) = -i\gamma^\mu \frac{\partial}{\partial x_\mu^\mathrm{E}} \frac{1}{6\pi^2 x_\mathrm{E}} &\left\{ \frac{1}{x_\mathrm{E}} F\left(\frac{1}{2};1,\frac{5}{2};1-\frac{m^2x_\mathrm{E}^2}{4}\right) - \frac{m^2x_\mathrm{E}}{4} \right\} \\ &\times F\left(\frac{3}{2};2,\frac{5}{2};1-\frac{m^2x_\mathrm{E}^2}{4}\right), \end{aligned}$$

and

$$\varphi_{p\ell}(x) = (1-i)\frac{u(p)}{\sqrt{2\omega}}\frac{1}{(2\pi)^{\frac{5}{2}}}\left\{B\left(\frac{1}{2},\frac{5}{4}\right)F\left(\frac{5}{4};\frac{1}{2},\frac{7}{4};-\frac{(px)_0^2}{4}\right)\right.$$
$$\left.+(px)_0 B\left(\frac{1}{2},\frac{7}{4}\right)F\left(\frac{7}{4};\frac{3}{2},\frac{9}{4};-\frac{(px)_0^2}{4}\right)\right\}. \qquad (345)$$

Here $px = p_0x_0 - \mathbf{px}$, $(px)_0 = \omega t - \mathbf{px}$, $\omega = \sqrt{m^2+\mathbf{p}^2}$, $x_{\mathrm{E}} = \sqrt{x_1^2 + \cdots + x_4^2}$, $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ and $F =_1 F_2(\alpha;\beta,\gamma;z)$ is the generalized hypergeometric function.

15.4. Appearance of a Superfield in the Radiation of the Torus Charge

Formulas (324) and (325) allow us to suppose that the torus charges radiate absorb complicated fields consisting of the nonlocal photon and nonlocal massless spinor fields. We call these fields superfields. However, as seen above in this mixed (or super) fields ratio ($\gamma = n_\gamma/n_{\text{photino}}$) of numbers of photons and massless spinors generated simultaneously by the charge of the torus is of the order of $\sim 1/\sqrt{R}$, where R is the big radius of the torus. On the other hand, from experimental data on testing local quantum electrodynamics it is well-known that if the electrical charge, say the electron possesses some innermost structure defined by the parameter R then its value is smaller or of the order of $R \sim 10^{-16}$cm. Therefore the ratio $\gamma \sim 10^8$. In this connection the following question arises. Why is it difficult to detect superpartners of usual particles? The simple answer is that they exist in very small portions with respect to usual particles, in superfield mixtures. In the language of supersymmetry, photon and photino fields can be formally present in the superfield form

$$\phi = \begin{pmatrix} \bar{\theta}\bar{\sigma}^\nu\theta A_\nu \\ \bar{\theta}\psi \end{pmatrix}, \qquad \phi^* = (\theta\sigma^\mu\bar{\theta}A_\mu, \quad \overline{\psi\theta}\theta\theta), \qquad (346)$$

where A_μ and ψ are the photon and the massless spinor fields, θ and $\bar{\theta}$ are Grassman variables. By means of these fields the Green function (324) in x space can be written in the form

$$D_\phi(x-y) = \int d^2\theta \int d^2\bar{\theta}\langle 0|T\{\phi^*(x)\phi(y)\}|0\rangle$$
$$= \eta^{\mu\nu}\eta_{\mu\nu}D_{\text{ph}}(x-y) + D^{\text{f}}(x-y). \qquad (347)$$

Here we have used the definitons

$$\sigma^\mu\bar{\sigma}^\nu = \eta^{\mu\nu} + 2\sigma^{\mu\nu}, \qquad \theta\sigma^{\mu\nu}\theta = 0,$$

and

$$\int d^2\theta\ \theta^2 = \int d^2\bar{\theta}\ \bar{\theta}^2 = 1. \tag{348}$$

Fourier components of $D_{\rm ph}(x)$ and $D^{\rm f}(x)$ are given by (324) and (325).

In the limit $R \to 0$ we can also write a formal equation

$$\iint d^2\theta d^2\bar{\theta} \begin{pmatrix} \Box \cdot \eta_\mu \theta \sigma^\mu \bar{\theta} & 0 \\ 0 & \frac{2}{3\pi} R\sqrt{\Box} \cdot \bar{\theta}\theta\theta \end{pmatrix} \begin{pmatrix} \bar{\theta} \cdot \bar{\sigma}^\nu \theta A^o_\mu \\ \bar{\theta}\psi^o \end{pmatrix} = 0. \tag{349}$$

Here A^o_μ and ψ^o are the local photon and local massless spinor fields. From this it follows

$$n_\nu \cdot \Box A^o_\nu + \frac{2}{3\pi} R \cdot \sqrt{\Box}\psi^o = 0, \tag{350}$$

where n_ν is an arbitrary unit constant vector belonging to the direction of the radiation.

In accordance with formulas (324) and (325) for the massive case $-p^2 \to m^2 - p^2$. Expressions (346), (347), (349), and (350) are reduced to the forms

$$\phi_{\rm m} = \begin{pmatrix} \theta\theta\varphi(x) \\ \bar{\theta}\Lambda(x) \end{pmatrix}, \qquad \phi^*_{\rm m} = (\varphi^*(x)\bar{\theta} \cdot \bar{\theta}, \Lambda(x)\bar{\theta}\theta\theta), \tag{351}$$

$$\begin{aligned} D^{\rm m}_\phi(x-y) &= \int d^2\theta \int d^2\bar{\theta} \langle 0|T\{\phi^*_{\rm m}(x)\phi_{\rm m}(y)\}|0\rangle \\ &= D^{\rm m}(x-y) + \int_{-1}^{1} d\lambda\, \rho(\lambda) S^{\rm m}(x-y, m\lambda), \end{aligned} \tag{352}$$

$$\int d^2\theta \int d^2\bar{\theta} \begin{pmatrix} (m^2 - \Box)\bar{\theta} \cdot \bar{\theta} & 0 \\ 0 & \frac{2}{3\pi i} R\sqrt{m^2 - \Box} \cdot \bar{\theta}\theta\theta \end{pmatrix} \begin{pmatrix} \theta\theta\varphi(x) \\ \bar{\theta}\Lambda(x) \end{pmatrix} = 0, \tag{353}$$

and

$$(m^2 - \Box)\varphi(x) - \frac{2}{3\pi} Ri \left(\sqrt{m^2 - \Box}\right) \Lambda(x) = 0. \tag{354}$$

where $\varphi(x)$ is the Klein–Gordon scalar particle of mass m and $\Lambda(x)$ is the square root particle defined by the expression

$$\Lambda(x) = \int_{-1}^{1} d\lambda\, \rho(\lambda)\psi(x, m\lambda) \tag{355}$$

Here $\psi(x, m\lambda)$ is the usual Dirac spinor with mass $m\lambda$. Functions $D^{\rm m}(x-y)$ and $S^{\rm m}(x-y, m\lambda)$ in (352) are nonlocal scalar and nonlocal spinor propagators, respectively.

Thus, we see that in physics of the torus superfields the inescapable appear. Of course, when the size of the torus tends to zero the superpartners of the photon and of usual scalar fields disappear in co-ordinating with the point of view of local quantum field theory. Electromagnetic interaction of the charged torus will be presented elsewhere.

APPENDIX: RADIATION IN LONGITUDINAL DIRECTIONS

In the above sections we have considered transverse radiations taking place in directions perpendicular to the plane in which extended charges are located. Strictly speaking, some longitudinal radiations may occur in directions parallel to the $z = 0$ plane. Now we study such possibilities. In this case, the Green function (295) generated by the ring potential (292) takes the form

$$\tilde{D}_{\rm r}^{\rm long}(p) = \int_{-\infty}^{\infty} dz \int_0^{\infty} d\rho\, \rho e^{-ip\rho} \int_0^{2\pi} d\varphi \left[\frac{U_{\rm r}(\rho, z)}{e}\right]. \tag{356}$$

Using series (293) with (294) one can see that the main module $n = 0$ of (356) is diverged. We regularize it as follows. Integration of the z variable gives

$$i_1 = 2\int_0^{\infty} dz[z^2 + (\rho + \ell)^2]^{-n-\frac{1}{2}} = \frac{\sqrt{\pi}\,\Gamma(n)}{\Gamma\left(n + \frac{1}{2}\right)}(\rho + \ell)^{-2n}, \tag{357}$$

where divergence is associated with the Γ-function $\Gamma(0) = \infty$.

One can easily calculate integration over the variable ρ:

$$\begin{aligned} i_2 = 4^n \ell^n \int_0^{\infty} d\rho\, \rho^{1+n}(\rho + \ell)^{-2n} e^{-ip\rho} &= \frac{1}{2i}\int_{-\gamma+i\infty}^{-\gamma-i\infty} \frac{d\beta}{\sin \pi\beta} \frac{1}{\Gamma(1 + 2\beta)} \\ &\times \left\{ I_1(p, \beta) - \frac{i}{1 + 2\beta} I_2(\beta, p)\right\}. \end{aligned} \tag{358}$$

We introduce regularization procedure into these integrals I_1 and I_2, which are defined by

$$\begin{aligned} I_1 &= 4^n \ell^{2\beta+2} \int_1^{\infty} dy\, y^{-\nu_1}(y - 1)^{\nu_1-\mu_1-1} \\ &= 4^n \ell^{2\beta+2} \lim_{\lambda\to 0} \int_1^{\infty} dy\, y^{\lambda-\nu_1}(y - 1)^{\nu_1-\mu_1-1}(\alpha y - 1)^{-\lambda} \end{aligned} \tag{359}$$

and

$$I_2 = 4^n \ell^{2\beta+3} \lim_{\lambda\to 0} \int_1^{\infty} dy\, y^{\lambda-\nu_2}(y - 1)^{\nu_2-\mu_2-1}(\alpha y - 1)^{-\lambda}. \tag{360}$$

Here $\nu_1 = \nu_2 = 2n$, $\mu_1 = n - 2\beta - 2$, $\mu_2 = n - 2\beta - 3$, and α is an arbitrary constant arisen from the regularization of the propagator in the longitudinal directions.

Taking into account the integrals (357)–(360) one gets

$$\tilde{D}_{\mathrm{r}}^{\mathrm{long}}(p) = 2\pi \left\{ \sum_{n=0}^{\infty} \left\{ \frac{(2n-1)!!}{2^n n!} \right\}^2 \right\} \frac{n}{\Gamma^2\left(n+\frac{1}{2}\right)} \lim_{\lambda\to 0} \alpha^{-1-\lambda} \frac{\Gamma(1+\lambda)}{\lambda} \cdot \frac{\ell^2}{2i}$$
$$\times \int_{-\gamma+i\infty}^{-\gamma-i\infty} \frac{d\beta}{\sin\pi\beta} \frac{(-p^2\ell^2)^\beta}{\Gamma(1+2\beta)} \left\{ \Gamma(-2\beta+n-1)\Gamma(n+2\beta+2) \right.$$
$$\times F(2n+1, n-2\beta-1; 2; \alpha^{-1}) + \frac{ip^2\ell}{\sqrt{-p^2}} \Gamma(-2\beta+n-2)$$
$$\left. \times \Gamma(n+2\beta+3) F(2n+1, n-2\beta-2; 2; \alpha^{-1}) \right\} \tag{361}$$

where we have used the hypergeometric function's property

$$\lim_{\gamma\to 0} \frac{1}{\Gamma(\gamma)} F(\alpha, \beta; \gamma; z) = \alpha \cdot \beta \cdot z F(\alpha+1, \beta+1; 2; z) \tag{362}$$

and the conditional notation $(2n-1)!! = 1$ for $n = 0, 1$. In (361), for the first and second terms $-1 < \gamma < -1/2$ and $-3/2 < \gamma < -1$, respectively. We see from (361) that the principle modules $n = 0$ in infinite series tends to zero.

Now we should remove regularization by taking limit $\lambda = 0$; however, due to an arbitrary choice of the parameter α one can suppose $\alpha = 1/\lambda$, and therefore whole integral (361) equals to zero. In other words, after removal of regularization its trace does not remain in the expression of the Green function. As a result, radiations by extended charges like ring, disk, holed coin and torus do not take place in the longitudinal directions.

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